PERMUTABLE RATIONAL FUNCTIONS*

BY

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INTRODUCTION

We investigate, in this paper, the circumstances under which two rational functions, $\Phi(z)$ and $\Psi(z)$, each of degree greater than unity,† are such that

$$\Phi[\Psi(z)] = \Psi[\Phi(z)].$$

A pair of functions of this type will be called *permutable*.

A memoir devoted to this problem has recently been published by Julia.‡ When $\Phi(z)$ and $\Psi(z)$ are polynomials, and are such that no iterate of one is identical with any iterate of the other, Julia shows how $\Phi(z)$ and $\Psi(z)$ can be obtained from the formulas for the multiplication of the argument in the functions e^z and $\cos z$. His other results are mainly of a qualitative nature, and deal with the manner in which $\Phi(z)$ and $\Psi(z)$ behave when iterated.

Certain of Julia's results have been announced independently by Fatou.§ Fatou's method is identical with that of Julia.

The method used in the present paper differs radically from that of Julia and Fatou, and leads to results of much greater precision. Its chief yield is the

THEOREM. *If the rational functions $\Phi(z)$ and $\Psi(z)$, each of degree greater than unity, are permutable, and if no iterate of $\Phi(z)$ is identical with any iterate of $\Psi(z)$,‖ there exist a periodic meromorphic function $f(z)$, and four numbers a, b, c and d, such that*

$$f(az+b) = \Phi[f(z)], \qquad f(cz+d) = \Psi[f(z)].$$

The possibilities for $f(z)$ are: any linear function of e^z, $\cos z$, $\wp z$; in the lemniscatic case ($g_3 = 0$), $\wp^2 z$; in the equianharmonic case ($g_2 = 0$), $\wp' z$

* Presented to the Society, February 24, 1923.

† The case in which one of the functions is linear will be met incidentally in § X.

‡ *Mémoire sur la permutabilité des fractions rationnelles,* Annales de l'Ecole Normale Supérieure, vol. 39 (1922), pp. 131–215.

§ Paris Comptes Rendus, Oct. 10, 1921.

‖ The condition that no common iterate exists is certainly satisfied if the degrees of the two functions are relatively prime.

and $\wp^3 z$. These are, essentially, the only periodic meromorphic functions which have rational multiplication theorems.*

The multipliers a and c must be such that if ω is any period of $f(z)$, $a\omega$ and $c\omega$ are also periods of $f(z)$.

If p represents the order of $f(z)$, that is, the number of times $f(z)$ assumes any given value in a primitive period strip or in a primitive period parallelogram, the products

$$b(1-e^{2\pi i/p}), \qquad d(1-e^{2\pi i/p})$$

must be periods of $f(z)$.

Finally,

$$(a-1)d-(c-1)b$$

must be a period of $f(z)$.

The condition that $\Phi(z)$ and $\Psi(z)$ have no iterate in common, can be replaced by one which is certainly not stronger, and which is satisfied, for instance, if there does not exist a rational function $\sigma(z)$, of degree greater than unity, such that

$$\Phi(z) = \varphi[\sigma(z)], \qquad \Psi(z) = \psi[\sigma(z)],$$

where $\varphi(z)$ and $\psi(z)$ are rational.

The existence of the periodic function $f(z)$ is demonstrated by a method which is almost entirely algebraic. It would be interesting to know whether a proof can also be effected by the use of the Poincaré functions employed by Julia.

Of the periodic functions listed above, the linear integral functions of e^z and of $\cos z$ are the only ones whose multiplication theorems will produce a pair of permutable polynomials. In all other cases, at least one of the functions $\Phi(z)$ and $\Psi(z)$ will be fractional. In § X we settle completely the case in which $\Phi(z)$ and $\Psi(z)$ are both polynomials, obtaining the

THEOREM. *If $\Phi(z)$ and $\Psi(z)$ are a pair of permutable polynomials (non-linear), which do not come from the multiplication theorems of e^z and* $\cos z$, *there exist a linear integral function $\lambda(z)$ and a polynomial*

$$G(z) = zR(z^r),$$

* These Transactions, vol. 23 (1922), p. 16.

where $R(z)$ is a polynomial, such that

$$\Phi(z) = \lambda^{-1}\{\varepsilon_1 G^{(\mu)}[\lambda(z)]\}, \qquad \Psi(z) = \lambda^{-1}\{\varepsilon_2 G^{(\nu)}[\lambda(z)]\},$$

where $G^{(i)}(z)$ represents the ith iterate of $G(z)$, and where ε_1 and ε_2 are rth roots of unity.

Thus, neglecting a linear transformation, $\Phi(z)$ and $\Psi(z)$ are iterates of the same polynomial, multiplied sometimes by roots of unity.

As to the permutable fractional functions which do not come from the multiplication theorems (and which therefore have an iterate in common), we give a method for constructing them, which, while it is not everything to be desired, still throws considerable light upon the functions under consideration. This method, which involves two types of operations, applies to all rational functions, integral or fractional.

Let $\Phi(z)$ and $\Psi(z)$ be two permutable rational functions. If there exist three rational functions, $\varphi(z)$, $\psi(z)$ and $\sigma(z)$, each of degree greater than unity, such that

$$\Phi(z) = \sigma[\varphi(z)], \qquad \Psi(z) = \sigma[\psi(z)],$$

and that $\varphi[\sigma(z)]$ is permutable with $\psi[\sigma(z)]$, we shall call the act of passing from $\Phi(z)$ and $\Psi(z)$ to $\varphi[\sigma(z)]$ and $\psi[\sigma(z)]$ an *operation of the first type.*

If $\Phi(z)$ and $\Psi(z)$ are permutable, it is evident that $\Phi[\Psi(z)]$ will be permutable both with $\Phi(z)$ and with $\Psi(z)$. We shall call the act of passing from $\Phi(z)$ and $\Psi(z)$ to $\Phi[\Psi(z)]$ and $\Phi(z)$, or to $\Phi[\Psi(z)]$ and $\Psi(z)$, an *operation of the second type.*

We show in § X that *if $\Phi(z)$ and $\Psi(z)$ do not come from the multiplication theorems of the periodic functions, there exists a linear function $\lambda(z)$ such that $\lambda^{-1}\Phi\lambda(z)$ and $\lambda^{-1}\Psi\lambda(z)$ can be obtained by repeated operations of the above two types, starting from a pair of functions*

$$zR(z^r), \qquad \varepsilon z R(z^r),$$

where $R(z)$ is a rational function, and where ε is an rth root of unity (sometimes unity itself).

For polynomials, only operations of the second type are necessary, and we obtain the explicit formulas given above. In the case of the fractional functions, however, operations of the first type are sometimes necessary, so that there exist permutable pairs of fractional functions which come neither from the multiplication theorems of the periodic functions, nor from the iteration of a function.

We have not succeeded thus far in determining all cases in which operations of the first type are possible. In fact, an illustration of the operations of that type, given in § X, will probably weaken any a priori conviction one might have to the effect that formulas as explicit as those stated above for polynomials can be found for the permutable fractional functions which have an iterate in common. It will not be inconceivable that too little order may prevail among the functions of that class for a complete enumeration of them to be possible.

The present paper is the outcome of efforts to solve, for fractional functions, the problem settled for polynomials in our paper *Prime and composite polynomials.**

1. Preliminaries

What we do principally in this section is to recall certain results proved in the above mentioned paper on prime and composite polynomials, on which the work in the present paper will be based.

Let $\varphi(z)$ and $\psi(z)$ be two rational functions of the respective degrees $r > 1$ and $s > 1$. Let $F(z) = \varphi[\psi(z)]$. We put

$$w = F(z) = \varphi(u), \qquad u = \psi(z).$$

It is easy to see the Riemann surface for $F^{-1}(w)$ is related to those for $\varphi^{-1}(w)$ and $\psi^{-1}(u)$. Suppose that, for $u = c$, $\psi^{-1}(u)$ has a critical point with a certain number of cycles. As $\varphi^{-1}(w)$ assumes no value more than once on its Riemann surface, $F^{-1}(w)$ will surely have a critical point for $w = \varphi(c)$. If the value c is assumed by a branch u_1 of $\varphi^{-1}(w)$ which is uniform in the neighborhood of $\varphi(c)$, those branches of $F^{-1}(w)$ for which $\psi(z) = u_1$ will be ramified at $\varphi(c)$ as the branches of $\psi^{-1}(u)$ are at c. If the value c is assumed by a cycle of p branches of $\varphi^{-1}(w)$, each cycle of $\psi^{-1}(u)$ at $u = c$ will lead to a cycle of $F^{-1}(w)$ at $\varphi(c)$ with p times as many sheets. If $\varphi^{-1}(w)$ has a critical point for $w = d$, and if none of the points $u = \varphi^{-1}(d)$ is a critical point of $\psi^{-1}(u)$, then each cycle of $\varphi^{-1}(w)$ at d leads to s cycles of the same number of sheets for $F^{-1}(w)$ at d.

We call the sum of the orders of the branch points of an algebraic function which are superimposed on each other at a given critical point the index of the function at the point. The sum of the indices of the inverse of a rational function of degree r, for all of its critical points, is $2r - 2$.

* These Transactions, vol. 23 (1922), p. 51.

It is easy to see that the index of $F^{-1}(w)$, at any critical point w_0 of $\varphi^{-1}(w)$, is at least s times the index of $\varphi^{-1}(w)$ at w_0. Also, if the index of $F^{-1}(w)$ at w_0 is q, and if w_0 is not a critical point of $\varphi^{-1}(w)$, then $\psi^{-1}(u)$ must have critical points whose affixes are values of $\varphi^{-1}(w)$ at w_0, and the sum of whose indices is q.

With respect to the group of monodromy of $F^{-1}(w)$, the rs branches of $F(w)$ break up into r systems of imprimitivity, such that if the branches

$$z_1, z_2, \ldots, z_s \tag{1}$$

constitute one of these systems, we have

$$\psi(z_1) = \psi(z_2) = \cdots = \psi(z_s).* \tag{2}$$

These r systems are said to be *determined* by $\psi(z)$.

Conversely, let $F(z)$ be any rational function of degree rs whose inverse has an imprimitive group. If (1) is a system of imprimitivity of the group of $F^{-1}(w)$, there exists a rational function $\psi(z)$, of degree s, for which (2) holds, and we have

$$F(z) = \varphi[\psi(z)],$$

where $\varphi(z)$ is a rational function of degree r. If another rational function determines the same systems of imprimitivity as $\psi(z)$, it is a linear function of $\psi(z)$.

Suppose that (1) can be broken up into smaller systems of imprimitivity, each containing t letters, and let $\sigma(z)$ be the rational function of degree t which determines these systems. Then $\psi(z)$ is a rational function of $\sigma(z)$.†

We shall deal next with five rational functions of degrees greater than unity; $\varphi_1(z)$ and $\varphi_2(z)$, each of degree r, $\psi_1(z)$ and $\psi_2(z)$, each of degree s, and $F(z)$. We suppose that

$$F(z) = \varphi_1[\psi_2(z)] = \psi_1[\varphi_2(z)].$$

We put $w = F(z)$, $u = \psi_2(z)$ and $v = \varphi_2(z)$, so that

$$w = \varphi_1(u) = \psi_1(v).$$

* Loc. cit., p. 54.

† Loc. cit., p. 55.

The function $\psi_2(z)$ determines r systems of imprimitivity of the group of $F^{-1}(w)$,

$$U_1, U_2, \ldots, U_r,$$

each containing s letters, while $\varphi_2(z)$ determines the s systems

$$V_1, V_2, \ldots, V_s,$$

each containing r letters. If w describes a closed path, the sets U are permuted like the branches of $\varphi_1^{-1}(w)$, and the sets V like the branches of $\psi_1^{-1}(w)$.

In what follows, we shall assume that each system U has exactly one letter in common with each system V. We see directly that if some substitution of the group of $F^{-1}(w)$ interchanges the letters of some set V_i among themselves, it interchanges the sets U with a substitution similar to that which it effects on the letters of V_i.

A point (w, u) on the Riemann surface of $\varphi_1^{-1}(w)$ for which w is a critical point of $\varphi_1^{-1}(w)$ or of $\psi_1^{-1}(w)$ (that is of $F^{-1}(w)$) will be called a *point* of $\varphi_1^{-1}(w)$. A point of $\psi_1^{-1}(w)$ is defined similarly. If p branches coalesce at a point, we shall say that the point is of *order* p. Thus a point of order p is a branch point of order $p-1$.

A point of order unity will be called a *simple point*. If $\varphi_1^{-1}(w)$ has a point of order p for $w = w_0$, and if $\psi_1^{-1}(w)$ has a critical point at w_0 where its branches undergo a substitution whose order is not a factor of p, the point of order p will be called an *A-point* of $\varphi_1^{-1}(w)$. An A-point of $\psi_1^{-1}(w)$ is defined similarly. If w_0 is a critical point of $\psi_1^{-1}(w)$, every simple point which $\varphi_1^{-1}(w)$ may have at w_0 is an A-point of $\varphi_1^{-1}(w)$.

Suppose that $\psi_1^{-1}(w)$ has a point of order p for $w = w_0$, and let the value of $\psi_1^{-1}(w)$ at the point be v_0. Suppose that as w makes a turn about w_0, the branches of $\varphi_1^{-1}(w)$ undergo a substitution S; S will be identity if w_0 is not a critical point of $\varphi_1^{-1}(w)$. Let w execute p turns about w_0, so that the branches of $\varphi_1^{-1}(w)$, and therefore the systems U, undergo the substitution S^p. Let v_i be one of those branches of $\psi_1^{-1}(w)$ which coalesce at the point of order p now under consideration. As w makes p turns about w_0, the value of v_i makes a single turn about v_0. Thus, by the p turns, the letters of V_i are interchanged among themselves; hence the substitution which these letters undergo is similar to S^p. This means that when v makes a turn about v_0, the branches of $\varphi_2^{-1}(v)$ undergo a substitution similar to S^p.

Thus, *a necessary and sufficient condition that* $\varphi_2^{-1}(v)$ *have a critical point at* v_0, *is that the value* v_0 *be assumed by* $\psi_1^{-1}(w)$ *at an A-point.* A similar result holds for $\psi_2^{-1}(u)$.

II. THE THREE SEQUENCES

We deal with the two permutable functions $\Phi(z)$ and $\Psi(z)$, of the respective degrees $m > 1$ and $n > 1$, and write

$$w = F(z) = \Phi[\Psi(z)] = \Psi[\Phi(z)],$$

or, more briefly,

$$F = \Phi\Psi = \Psi\Phi.$$

The greatest single source of work in this paper is the possibility of the existence of a rational function $\sigma_0(z)$, of degree greater than unity, such that

$$\Phi = \varphi_0\sigma_0, \qquad \Psi = \psi_0\sigma_0,$$

where $\varphi_0(z)$ and $\psi_0(z)$ are rational (even linear). Wherever the contrary is not stated, we shall assume that such a $\sigma_0(z)$ exists.

With a view towards securing later a sharp separation of permutable pairs of functions into two classes, we establish now a definite method for selecting $\sigma_0(z)$. We proceed as follows. As $F = \Phi\Psi$, the function $\Psi(z)$ determines m systems of imprimitivity of the group of $F^{-1}(w)$, each containing n letters. Also, if $\Psi = \psi_0\sigma_0$, the function $\sigma_0(z)$ determines systems of imprimitivity of the group of $F^{-1}(w)$. Two branches, z_1 and z_2, will be in the same one of the systems determined by $\sigma_0(z)$ if $\sigma_0(z_1) = \sigma_0(z_2)$. But as $\Psi(z_1) = \Psi(z_2)$ in this case, z_1 and z_2 are both in one system determined by $\Psi(z)$. Hence every system determined by $\sigma_0(z)$ is contained in a system determined by $\Psi(z)$, so that each system determined by $\Psi(z)$ is composed of one or more systems determined by $\sigma_0(z)$. A similar fact is true of $\Phi(z)$.

Thus, given any system determined by $\Phi(z)$, there is at least one system determined by $\Psi(z)$ with which it has more than one letter in common. Now, it is a simple consequence of the elementary notions on imprimitivity that if a group has two sets of systems of imprimitivity, there exists a number t_0, such that if a system of the first set has at least one letter in common with a system of the second set, it has precisely t_0 letters in common with it. These systems of t_0 letters are themselves systems of imprimitivity.

We shall suppose, in what follows, that $\sigma_0(z)$ is so taken that it determines the systems of imprimitivity of t_0 letters just shown to exist. This determines

$\sigma_0(z)$ to within a linear function, and the particular disposition made in regard to this linear function is of no importance for what follows.

We see now that there exists no rational function $\beta(z)$, of degree greater than unity, such that

$$\varphi_0 = \zeta\beta, \qquad \psi_0 = \xi\beta,$$

where $\zeta(z)$ and $\xi(z)$ are rational. If there did, each system of the group of $F^{-1}(w)$ determined by $\Phi(z)$ would have more than t_0 letters in common with some system determined by $\Psi(z)$.

We have

$$\varphi_0\,\sigma_0\,\psi_0\,\sigma_0 = \psi_0\,\sigma_0\,\varphi_0\,\sigma_0,$$

so that

(3) $$\varphi_0\,\sigma_0\,\psi_0 = \psi_0\,\sigma_0\,\varphi_0.$$

Let the degrees of $\varphi_0(z)$, $\psi_0(z)$, $\sigma_0(z)$ be r_0, s_0, t_0, respectively. We represent each member of (3) by G. The function $\sigma_0\,\psi_0$ determines r_0 systems of imprimitivity of the group of G^{-1},* each containing $s_0\,t_0$ letters. Also, $\sigma_0\,\varphi_0$ determines s_0 systems of imprimitivity of the group of G^{-1}, each containing $r_0\,t_0$ letters. The $s_0\,t_0$ letters in any system determined by $\sigma_0\,\psi_0$ are distributed among the s_0 systems determined by $\sigma_0\,\varphi_0$. Hence given any system determined by $\sigma_0\,\psi_0$, there is some system determined by $\sigma_0\,\varphi_0$ with which it has at least t_0 letters in common. Thus, by what goes before, there exists a number $t_1 \geqq t_0$, such that if a system determined by $\sigma_0\,\psi_0$ has at least one letter in common with a system determined by $\sigma_0\,\varphi_0$, the two systems have precisely t_1 letters in common. The systems determined by $\sigma_0\,\psi_0$ and by $\sigma_0\,\varphi_0$ break up into a third set of systems of imprimitivity, which are determined by a function $\sigma_1(z)$, of degree t_1. Also, we have

$$\sigma_0\,\varphi_0 = \varphi_1\,\sigma_1, \qquad \sigma_0\,\psi_0 = \psi_1\,\sigma_1,$$

where $\varphi_1(z)$ and $\psi_1(z)$ are rational functions of the respective degrees $r_1 \leqq r_0$ and $s_1 \leqq s_0$. Furthermore, there exists no rational $\beta(z)$ of degree greater than unity such that

$$\varphi_1 = \zeta\beta, \qquad \psi_1 = \xi\beta,$$

where $\zeta(z)$ and $\xi(z)$ are rational. Finally, by (3),

$$\varphi_0\,\psi_1\,\sigma_0 = \psi_0\,\varphi_1\,\sigma_0,$$

* If φ_0 is linear, we have to consider all of the branches of G^{-1} as forming a single system of imprimitivity.

so that

$$\varphi_0 \psi_1 = \psi_0 \varphi_1 .$$

We now subject the permutable functions $\varphi_1 \sigma_1$ and $\psi_1 \sigma_1$* to the treatment given above to $\Phi(z)$ and $\Psi(z)$. What follows is plain. There exist three sequences

(A) $\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_i, \ldots .$

(B) $\psi_0, \psi_1, \psi_2, \ldots, \psi_i, \ldots .$

(C) $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_i, \ldots .$

the degrees of the functions in the first two sequences being non-increasing, those of the functions in the third sequence non-decreasing, and the following relations holding for every i:

$$\varphi_i \sigma_i \psi_i \sigma_i = \psi_i \sigma_i \varphi_i \sigma_i,$$

$$\sigma_i \varphi_i = \varphi_{i+1} \sigma_{i+1}, \qquad \sigma_i \psi_i = \psi_{i+1} \sigma_{i+1},$$

$$\varphi_i \psi_{i+1} = \psi_i \varphi_{i+1};$$

furthermore, for no i does a function $\beta(z)$ of degree greater than unity exist such that

$$\varphi_i = \zeta\beta, \qquad \psi_i = \xi\beta,$$

where $\zeta(z)$ and $\xi(z)$ are rational.

For the case in which no $\sigma_0(z)$ exists, we define the sequences (A) and (B) by the equations

$$\varphi_i = \Phi, \qquad \psi_i = \Psi.$$

and all facts proved for (A) and (B) when (C) exists will hold also in this case.

From the monotonic character of the degrees of the functions in the three sequences, and from the fact that the degree of every $\sigma_i(z)$ is not greater than the smaller of m and n, we derive the

CONCLUSION. *There exist a subscript i_0, and three integers r, s and t, such that, for $i \geqq i_0$, $\varphi_i(z)$, $\psi_i(z)$ and $\sigma_i(z)$ are of the respective degrees r, s and t.*

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* We have from (3), $\sigma_0 \varphi_0 \sigma_0 \psi_0 = \sigma_0 \psi_0 \sigma_0 \varphi_0$.

From this point on, until we come to the last section of our paper, we shall work under the assumption that r and s each exceed unity. We assume, that is, that for no i is one of the permutable functions $\varphi_i \sigma_i$ and $\psi_i \sigma_i$ a rational function of the other. This understood, we prove the important

LEMMA. *There exist a subscript i_1, and two integers $h \leqq 4$ and $k \leqq 4$, such that, for $i \geqq i_1$, every $\varphi_i(z)$ is of degree r, while its inverse has precisely h critical points; and every $\psi_i(z)$ is of degree s, while its inverse has precisely k critical points.*

We write

$$w = F(z) = \varphi_i[\psi_{i+1}(z)] = \psi_i[\varphi_{i+1}(z)],$$

and assume that $i \geqq i_0$, so that $\varphi_i(z)$ and $\varphi_{i+1}(z)$ are each of degree r, and $\psi_i(z)$ and $\psi_{i+1}(z)$ each of degree s. As there exists no non-linear rational $\beta(z)$ such that $\varphi_{i+1} = \zeta\beta$, $\psi_{i+1} = \xi\beta$, where $\zeta(z)$ and $\xi(z)$ are rational, each of the systems of imprimitivity of the group of $F^{-1}(w)$ determined by $\varphi_{i+1}(z)$ has precisely one letter in common with each system determined by $\psi_{i+1}(z)$. Hence we can employ the notion of the A-point.

Suppose that φ_i^{-1} has g critical points, $w_1, \ldots, w_g$. We seek a lower bound for the sum of the indices of ψ_i^{-1} at these g points. That sum equals $gs - j$, where j is the number of points which ψ_i^{-1} has at $w_1, \ldots, w_g$. If p of these points are simple points, the other $j-p$ are at least of order 2, and we have

$$2(j-p)+p \leqq gs, \tag{4}$$

so that $j \leqq (gs+p)/2$, and the sum of the indices of ψ_i^{-1} at $w_1, \ldots, w_g$ is at least $(gs-p)/2$. We observe that if one of the j points is of order greater than 2, or if there are fewer than p simple points, (4) must be an inequality, and the sum of the indices of ψ_i^{-1} at $w_1, \ldots, w_g$ will exceed $(gs-p)/2$.

Suppose now that φ_{i+1}^{-1} has fewer than g critical points. Then ψ_i^{-1} must have fewer than g simple points at $w_1, \ldots, w_g$, for every such simple point of ψ_i^{-1} is an A-point, and yields a critical point of φ_{i+1}^{-1}. Hence the sum of the indices of ψ_i^{-1} at $w_1, \ldots, w_g$ exceeds $(gs-g)/2$ and we have

$$\frac{gs-g}{2} < 2s-2.$$

so that $g < 4$. Thus φ_i^{-1} has three critical points, and φ_{i+1}^{-1} has two.

At each of the two critical points of φ_{i+1}^{-1}, its r branches must be permuted in a single cycle. From the manner in which the critical points of φ_{i+2}^{-1} depend on those of φ_{i+1}^{-1}, we see that, at every critical point of φ_{i+2}^{-1}, its branches undergo a substitution which is a power of a cyclic substitution in r letters. Such a substitution must be regular, that is, it displaces every letter, and the order of its cycles are all equal. It follows that, for $j > i$, the critical points of every φ_j^{-1} have regular substitutions. Hence at every critical point of $\varphi_j^{-1} (j > i)$, all of the branches of φ_j^{-1} are permuted, so that the index of φ_j^{-1}, at each of its critical points, is at least $r/2$. As the sum of the indices of every φ_j^{-1} is $2r - 2$, every φ_j^{-1} has either two critical points or three.

If a $\varphi_j^{-1} (j > i)$ has three critical points, it cannot have one at which its branches are permuted in a single cycle; in that case the sum of its indices would exceed $2r - 2$. Hence φ_{j+1}^{-1} must also have three critical points, for φ_j^{-1} cannot transmit to φ_{j+1}^{-1} a critical point with a single cycle.

We see now that if φ_{i+1}^{-1} has fewer critical points than φ_i^{-1}, then, either each φ_j^{-1} has two critical points for $j > i$, or else, for j sufficiently large, each φ_j^{-1} has three critical points. It remains only to settle the case in which, for every $i > i_0$, φ_{i+1}^{-1} has at least as many critical points as φ_i^{-1}. In this case, since φ_i^{-1} cannot have more than $2r - 2$ critical points, is is evident that an h exists, such that, for i sufficiently large, each φ_i^{-1} has precisely h critical points. When φ_i^{-1} and φ_{i+1}^{-1} have an equal number of critical points, we find that the g of the preceding page does not exceed 4. Hence $h \leqq 4$.

The argument for $\varphi_i(z)$ holds also for $\psi_i(z)$, and the lemma is proved.

III. The critical points of φ_i^{-1} and ψ_i^{-1}

From this point on, every subscript i will be understood to be not less than the i_1 of the preceding lemma. We assume also that $h \geqq k$; if this is not so at the start, we need only interchange the designations of $\Phi(z)$ and $\Psi(z)$. Let $h = 4$, and let the critical points of φ_i^{-1} be $w_1, \ldots, w_4$. If ψ_i^{-1} has $p \leqq 4$ simple points at $w_1, \ldots, w_4$, the sum of the indices of ψ_i^{-1} at $w_1, \ldots, w_4$ is at least $(4s - p)/2$; if one of the points of ψ_i^{-1} at $w_1, \ldots, w_4$ is of order greater than 2, this lower bound must be increased. We must thus have

$$\frac{4s - p}{2} \leqq 2s - 2. \tag{5}$$

If p were less than 4, or if ψ_i^{-1} had a point of order greater than 2 at $w_1, \ldots, w_4$ (5) could not hold. Hence $p = 4$, and those points of ψ_i^{-1} at $w_1, \ldots, w_4$ which are not simple are all of order 2. Furthermore, it is clear that ψ_i^{-1} can have no critical points other than $w_1, \ldots, w_4$.

We shall see below that the points of φ_i^{-1} are also all of order 2, except four which are simple.

Let $h = 3$, and let the critical points of φ_i^{-1} be a, b, and c. We show first that ψ_i^{-1} has no critical points other than a, b, c.

Suppose first that $s > 3$. If a were not a critical point of ψ_i^{-1}, ψ_i^{-1} would have at least four simple points at a, so that φ_{i+1}^{-1} would have at least four critical points. Hence a, and similarly b and c, are critical points of ψ_i^{-1}. As $k \leqq 3$, ψ_{i+1}^{-1} can have no other critical points.

Let $s = 3$. If a is not a critical point of ψ_i^{-1}, ψ_i^{-1} must have no simple point at b or at c. This means that the branches of ψ_i^{-1} are permuted in a single cycle at b and at c, so that b and c are the only critical points of ψ_i^{-1}.

Let $s = 2$; ψ_i^{-1} has just two branch points, which are both simple. If one or both of these were not placed at a, b, c, at least two of these latter points would not be critical points of ψ_i^{-1}, and ψ_i^{-1} would have at least four A-points.

Thus, when $h = 3$, ψ_i^{-1} has no critical points other than a, b, c. When $h = 3$, we must have $r \geqq 3$. But we have seen above that when $h = 3$ and $k = 2$, we have $s \leqq 3$. Hence, in the case of $h = 3$, it is legitimate to assume that $r \geqq s$. In §§ V, VII, VIII, IX, which deal with the case of $h = 3$, it will be understood, unless otherwise stated, that $r \geqq s$.

Suppose that $h = 3$, and that at the critical points a, b, c, of φ_i^{-1}, the branches of φ_i^{-1} undergo substitutions of orders $\alpha_i, \beta_i, \gamma_i$, respectively. The function ψ_i^{-1} must have precisely three A-points. Let x be the sum of the orders of the A-points of ψ_i^{-1} at a, y at b and z at c. The orders of those points of ψ_i^{-1} at a which are not A-points are divisible by α_i. Hence their number is at most $(s - x)/\alpha_i$. Thus, the total number of points of ψ_i^{-1} at a, b, c is at most

$$\frac{s-x}{\alpha_i} + \frac{s-y}{\beta_i} + \frac{s-z}{\gamma_i} + 3. \tag{6}$$

As ψ_i^{-1} has no critical points other than a, b, c, the sum of its indices at a, b, c is $2s - 2$. This means that ψ_i^{-1} has $s + 2$ points at a, b, c. Thus the expression (6) is at least $s + 2$, and we find, directly,

$$\frac{1}{\alpha_i} + \frac{1}{\beta_i} + \frac{1}{\gamma_i} \geq 1 - \frac{1}{s} + \frac{1}{s}\left(\frac{x}{\alpha_i} + \frac{y}{\beta_i} + \frac{z}{\gamma_i}\right). \tag{7}$$

In particular, the first member of (7) exceeds $(1 - 1/s)$. Suppose that ψ_i^{-1} has at a an A-point of order g. This A-point gives rise to a critical point of φ_{i+1}^{-1} at which the branches of φ_{i+1}^{-1} undergo a substitution similar to the gth power of the substitution which the branches of φ_i^{-1} undergo at a. The order, call it α_{i+1}, of the substitution at this critical point of φ_{i+1}^{-1} is α_i divided by the greatest common divisor of α_i and g. Certainly, then, $1/\alpha_{i+1}$ is not greater than g/α_i. It is easy now to see that

$$\frac{x}{\alpha_i}+\frac{y}{\beta_i}+\frac{z}{\gamma_i}\geqq\frac{1}{\alpha_{i+1}}+\frac{1}{\beta_{i+1}}+\frac{1}{\gamma_{i+1}},$$

so that

$$\frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}\geqq 1-\frac{1}{s}+\frac{1}{s}\left(\frac{1}{\alpha_{i+1}}+\frac{1}{\beta_{i+1}}+\frac{1}{\gamma_{i+1}}\right). \tag{8}$$

But the quantity in parentheses in the second member of (8) exceeds $1-1/s$, the lower bound secured above for the first member. Hence

$$\frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}>1-\frac{1}{s}+\frac{1}{s}\left(1-\frac{1}{s}\right)=1-\frac{1}{s^2}.$$

This gives a new lower bound for the quantity in parentheses in (8), which, when substituted, shows that the first member is not less than $1-1/s^3$. As this process may be repeated indefinitely, we arrive at the

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$$\frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}\geqq 1. \tag{9}$$

When $h=4$, the same procedure leads easily to the result that the sum of the reciprocals of the orders of the substitutions at the four critical points of φ_i^{-1} is at least 2. As each order is at least 2, each order must be precisely 2. Thus, all points of φ_i^{-1} which are not simple are of order 2. It follows, as at the beginning of this section, that φ_i^{-1} has precisely four simple points.

We suppose finally that $h=2$. Two cases have to be considered:

(*a*) Either r or s exceeds 2. In this case, φ_i^{-1} and ψ_i^{-1} must have the same two critical points, else one of them would have more than two simple points, and these would be A-points. At both critical points, the branches of both functions are permuted in single cycles.

(*b*) $r=s=2$. In this case, φ_i^{-1} and ψ_i^{-1} cannot have the same two critical points, else neither would have an A-point. If they had no critical point in common, each would have four A-points. Hence we may assume that φ_i^{-1} has simple branch points at each of two points a and b, and that ψ_i^{-1} has one simple branch point at a, and one at a third point c.

In what follows, special attention has to be given to certain cases in which s is small. A device which would permit us to assume that r and s are arbitrarily large, one based, for instance, on replacing the permutable functions by iterates of themselves, would eliminate many painful paragraphs.

IV. THE MULTIPLICATION FORMULAS FOR e^z; THE POWERS OF z

We consider the case in which $h = 2$, and in which r and s are not both 2.

In this case, φ_i^{-1} and ψ_i^{-1} each have the same two critical points, a_i and b_i. Both branch points of either function are A-points of that function. Also, the values a_{i+1} and b_{i+1} which φ_i^{-1} assumes at its A-points are the same two values which ψ_i^{-1} assumes at its A-points, for these are the affixes of the critical points of φ_{i+1}^{-1} and ψ_{i+1}^{-1}.*

Turning now to the sequence (C), we shall prove that σ_i^{-1} has no critical points other than a_{i+1} and b_{i+1}. Suppose that σ_i^{-1} has such critical points other than a_{i+1} and b_{i+1}, and let m_i be the sum of its indices at those additional critical points. As $m_i \leqq 2t - 2$, there is an i for which m_i is a maximum. Suppose that it is this i with which we are dealing. Consider the relation

$$\sigma_i \varphi_i = \varphi_{i+1} \sigma_{i+1}. \tag{10}$$

From a remark made at the beginning of § I, we see that the sum of the indices of $\varphi_i^{-1} \sigma_i^{-1}$ at the critical points of σ_i^{-1} other than a_{i+1} and b_{i+1} is at least rm_i. Turning to the second member of (10) we see that σ_{i+1}^{-1} has critical points whose affixes are not values of φ_{i+1}^{-1} at a_{i+1} and b_{i+1}, that is, not a_{i+2} or b_{i+2}, at which the sum of its indices is at least rm_i. This contradicts the assumption that m_i is a maximum.

Thus the inverses of $\varphi_i \sigma_i$ and $\psi_i \sigma_i$ have the two critical points a_i, b_i, at which all of their branches are permuted in single cycles. From this, and from the relation

$$\sigma_i \psi_i = \psi_{i+1} \sigma_{i+1},$$

it follows that the two critical points of ψ_i^{-1} are the values which σ_i^{-1} assumes at a_{i+1} and b_{i+1}. Hence the inverse of

$$F = \varphi_i \sigma_i \psi_i \sigma_i$$

* It is not necessary that φ_i^{-1} and ψ_i^{-1} should assume the same value at a particular A-point.

has only the two critical points a_i and b_i. Also, from the condition of permutability, we see that the two values a_i and b_i are assumed by the inverses of both $\varphi_i \sigma_i$ and $\psi_i \sigma_i$ at a_i and b_i. These same values are assumed by F^{-1} at a_i and b_i.

The relation

$$\varphi_i \sigma_i \psi_i \sigma_i = \sigma_{i-1} \varphi_{i-1} \sigma_{i-1} \psi_{i-1}$$

shows that σ_{i-1} has no critical point other than a_i and b_i, and that the inverse of $\varphi_{i-1} \sigma_{i-1} \psi_{i-1}$ has no critical points other than a_{i-1} and b_{i-1}, the values (so we designate them) which σ_{i-1}^{-1} assumes at its critical points. Hence the inverse of the function

$$\varphi_{i-1} \sigma_{i-1} \psi_{i-1} \sigma_{i-1} = \psi_{i-1} \sigma_{i-1} \varphi_{i-1} \sigma_{i-1}$$

has no critical points other than a_{i-1} and b_{i-1}.

Thus the inverses of $\varphi_{i-1} \sigma_{i-1}$ and $\psi_{i-1} \sigma_{i-1}$ have a_{i-1} and b_{i-1} as their only critical points, and the values assumed by the inverses at their critical points are a_{i-1} and b_{i-1}.

Continuing thus, we find that the inverses of the original permutable functions

$$\Phi = \varphi_0 \sigma_0, \qquad \Psi = \psi_0 \sigma_0$$

have two critical points a_0 and b_0, and that the values of Φ^{-1} and Ψ^{-1} at their critical points are a_0 and b_0.

The statement just made in regard to $\Phi(z)$ and $\Psi(z)$ holds also if no σ_0 exists; it is only necessary to discard those parts of the proof which involve a σ.

Let $\lambda(z)$ be any linear function such that $\lambda(a_0) = 0$, and $\lambda(b_0) = \infty$. Let

$$\Phi_1 = \lambda \Phi \lambda^{-1}, \qquad \Psi_1 = \lambda \Psi \lambda^{-1}.$$

Then Φ_1^{-1} and Ψ_1^{-1} have the two critical points 0 and ∞, and their values at these points are 0 and ∞.

It follows that

$$\Phi_1(z) = \eta z^p, \qquad \Psi_1(z) = \varepsilon z^q,$$

where $p = \pm m$, and $q = \pm n$. If we multiply $\lambda(z)$ by a suitable constant, we will have $\eta = 1$. The condition of permutability then gives $\varepsilon^{p-1} = 1$.

Thus all pairs of permutable functions, of the type considered in this section, are found by transforming with a linear function the permutable pair z^p and εz^q where p and q are positive or negative integers, and where $\varepsilon^{p-1} = 1$.

V. The multiplication formulas for $\cos z$

This section and §§ VII, VIII, IX will handle the cases in which $h = 3$. We saw in § III that it is permissible to assume, when $h = 3$, that $r \geqq s$; where the contrary is not stated, this assumption will be understood to hold.

In all cases for which $h = 3$, we represent the critical points of φ_i^{-1} by a_i, b_i, c_i, and the orders of the substitutions which the branches of φ_i^{-1} undergo at a_i, b_i, c_i by $\alpha_i, \beta_i, \gamma_i$, respectively. The orders of the substitutions which the branches of ψ_i^{-1} undergo at a_i, b_i, c_i we denote by $\alpha_i', \beta_i', \gamma_i'$, respectively.

We consider in this section the case in which, for some $i \geqq i_1$, two of the orders $\alpha_i, \beta_i, \gamma_i$, say α_i and β_i, equal 2. The i used in this section is supposed to be of the type just described, and stays fixed throughout our work.

Consider first the case in which $k = 3$. We know that the critical points of ψ_i^{-1} are a_i, b_i, c_i. We are going to show that $\alpha_i' = \beta_i' = 2$.

As $r \geqq 3$, and as the branch points of φ_i^{-1} at a_i and b_i are all simple, φ_i^{-1} has at least four points (together) at a_i and b_i. If α_i' and β_i' both exceeded 2, all of these points would be A-points, whereas we know that φ_i^{-1} has only three A-points. We may suppose thus that $\alpha_i' = 2$.

If $r > 6$, φ_i^{-1} has at least four points at b_i, so that β_i' is also 2. The cases in which $r \leqq 6$, which create a type of nuisance of which there will be more later, we have to examine in detail.

Suppose that $\beta_i' > 2$. If $r = 6$, φ_i^{-1} must have three branch points (simple), at b_i; if it had fewer, it would have four or more A-points at b_i. Hence, $\alpha_{i+1}', \beta_{i+1}', \gamma_{i+1}'$, the orders of the substitutions at the critical points of ψ_{i+1}^{-1}, are all equal. Thus, by (9), $3/\alpha_{i+1} \geqq 1$, and α_{i+1} is either 3 or 2.

Suppose that $\alpha_{i+1}' = 3$. As the branch points of φ_i^{-1} at b_i are all simple, the substitutions at the critical points of ψ_{i+1}^{-1} are all similar to the square of the substitution which the branches of ψ_i^{-1} undergo at b_i, so that $\beta_i' = 6$, or $\beta_i' = 3$. If $\beta_i' = 6$, we see, remembering that $r \geqq s$, that $s = 6$. This produces the absurdity that the sum of the indices of ψ_{i+1}^{-1} is 12, or more than $2s - 2$. If $\beta_i' = 3$, we have $s \geqq 3$. If $s > 3$, then, since β_i is prime to β_i', ψ_i^{-1} has at least two A-points at b_i, so that φ_{i+1}^{-1} must have at least two critical points at which its branch points are simple. This contradicts the fact that $\alpha_{i+1}', \beta_{i+1}', \gamma_{i+1}'$ all equal 3, for we saw above that one of them must be 2 if two of $\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}$ are 2. Finally, if $s = 3$, ψ_i^{-1} has an A-point at a_i, as well as at b_i, so that the argument just used applies.

Suppose that $\alpha_{i+1}' = 2$. The index of ψ_{i+1}^{-1} cannot exceed $s/2$ at any critical point. Hence,

$$\frac{3s}{2} \geqq 2s - 2.$$

or $s \leqq 4$. We cannot have $s = 3$, as ψ_{i+1}^{-1} would have only three simple branch points in that case. If $s = 4$, there must be two simple branch points at each critical point of ψ_{i+1}^{-1}. Hence ψ_{i+1}^{-1} would have an even number of A-points, whereas it must have three. This completes the proof that $\beta_i' = 2$ when $r = 6$.

When $r = 5$, φ_i^{-1} has at least one A-point at a_i. If $\beta_i' > 2$, it would have at least three at b. Thus $\beta_i' = 2$.

Suppose that $r = 4$ and that $\beta_i' = 2$. We must have $s \leq 4$, so that $\beta_i' \leqq 4$.

If φ_i^{-1} has two simple branch points both at a_i and at b_i, we may suppose that the substitutions at these points are (12) (34) and (13) (24) respectively. Hence the substitution at c_i is (14) (23). This, as seen above, leads to the absurdity that φ_i^{-1} has an even number of A-points.

Thus φ_i^{-1} must have two simple points, either at a_i or at b_i. As φ_i^{-1} has at least two A-points at b_i when $\beta_i' > 2$, the simple points cannot be at a_i, else φ_i^{-1} would have at least four A-points. Then φ_i^{-1} has a simple branch point and two simple points at b_i. Suppose that $s = 4$. If β_i' were 4, ψ_{i+1}^{-1} would have two critical points of index 3, and a third critical point, which is impossible, since the sum of its indices is 6. If β_i' were 3, we would have $\alpha_{i+1}' = \beta_{i+1}' = \gamma_{i+1}' = 3$, and also φ_{i+1}^{-1} would have two critical points with only simple branch points, which come from the two A-points of ψ_i^{-1} at b_i. This was proved impossible above. If $s = 3$, and $\beta_i' = 3$, ψ_{i+1}^{-1} would have three critical points of index 2, an impossibility.

Suppose that $r = 3$ and that $\beta_i' > 2$. We find the absurdity that ψ_{i+1}^{-1} has two critical points of index 2 which come from b_i, and a third critical point which comes from a_i.

We have proved that $\alpha_i' = \beta_i' = 2$.

We shall now show that φ_i^{-1} and ψ_i^{-1} each have two simple points (two in all), at a_i and b_i. Consider φ_i^{-1} for instance. If r is odd, the two simple points certainly exist. In the case where r is even, if there were no such simple points, the sum of the indices of φ_i^{-1} at a_i and b_i would be r. Hence the index at c_i would be $r - 2$, and φ_i^{-1} would have precisely two points at c_i. These would be the only possible A-points of φ_i^{-1}, whereas there have to be three.

It follows from the above that, at c_i, all of the branches of φ_i^{-1} are permuted in a single cycle. The same is true of ψ_i^{-1}.

We examine now the case in which $h = 3$, $\alpha_i = \beta_i = 2$, and $k = 2$.

As seen in § III, s must be 3 or 2. If s were 3, the two branch points of ψ_i^{-1} would be points of order 3. At least one of them would be situated either at a_i or at b_i, and would be an A-point of ψ_i^{-1}. Also ψ_i^{-1} would have 3 A-points at that critical point of φ_i^{-1} which is not a critical point of ψ_i^{-1}.

Thus $s = 2$. If a_i and b_i were both critical points of ψ_i^{-1}, ψ_i^{-1} would have only two A-points. Hence we may assume that the critical points of ψ_i^{-1} are b_i and c_i.

We shall prove that r is odd, from which it will follow that φ_i^{-1} has one simple point at a_i, one at b_i, and that its branches are permuted in a single cycle at c_i.

Suppose, contrarily, that r is even. We know that φ_i^{-1} cannot have more than two simple points (in all) at a_i and b_i; if it did, its index at c_i would exceed $r-1$. Then φ_i^{-1} can have no simple point at a_i. If it had one, it would have two, and as ψ_i^{-1} has two A-points at a_i, φ_{i+1}^{-1} would have two critical points with simple branch points and with four simple points, a condition as impossible for φ_{i+1}^{-1} as for φ_i^{-1}.

It follows that φ_{i+1}^{-1} has two critical points with simple branch points and no simple points. Furthermore, it is permissible to replace $i+1$ by i, and to assume that φ_i^{-1} has no simple point at a_i or at b_i.

This understood, it follows that φ_i^{-1} has just two points at c_i, which must both be A-points. Consider the relation

$$F = \varphi_i\,\psi_{i+1} = \psi_i\,\varphi_{i+1}. \tag{11}$$

From $F=\varphi_i\,\psi_{i+1}$, we see that the two values which F^{-1} assumes at c_i are the two values which ψ_{i+1}^{-1} assumes at its critical points. As only one branch point of ψ_{i+1}^{-1} is an A-point, only one of the values of F^{-1} at c_i can be an affix of a critical point of φ_{i+2}^{-1}, or, a fortiori, of ψ_{i+2}^{-1}. Now φ_{i+1}^{-1} has two critical points with simple branch points and no simple points, at the points whose affixes are the values of ψ_i^{-1} at a_i. Hence at the point whose affix is the value of ψ_i^{-1} at c_i (c_i is an A-point of ψ_i^{-1}), φ_{i+1}^{-1} has two points, which, as in the case of φ_i^{-1}, must be A-points. This, since $F=\psi_i\,\varphi_{i+1}$, entails the contradiction that the values of F^{-1} at c_i are both affixes of critical points of ψ_{i+2}^{-1}. Thus r is odd.

When $h=k=3$, the values of φ_i^{-1} and ψ_i^{-1} at their simple points at a_i and b_i must be the same, namely, the affixes of the critical points with simple branch points of φ_{i+1}^{-1} and ψ_{i+1}^{-1}. Similarly, φ_i^{-1} and ψ_i^{-1} must have the same single value at c_i.

Suppose that, when $k=2$, $\varphi_i^{-1}(c_i)=c_{i+1}$ and $\psi_i^{-1}(c_i)=c'_{i+1}$. We shall show that $c_{i+1}=c'_{i+1}$.

We know that ψ_{i+1}^{-1} has a critical point at c_{i+1}, and that φ_{i+1}^{-1} has a critical point at c'_{i+1} at which its sheets are permuted in a single cycle. Let c'_{i+2} and c_{i+2} be the values of ψ_{i+1}^{-1} and φ_{i+1}^{-1} at c_{i+1} and c'_{i+1} respectively. From (11), since F^{-1} has only one value at c_i, we have $c_{i+2}=c'_{i+2}$. But since φ_{i+1}^{-1} has an A-point where its branches are permuted in a single cycle, c_{i+2} is a critical point of ψ_{i+2}^{-1}, and hence of φ_{i+2}^{-1}. Thus ψ_{i+1}^{-1} must have an A-point at its critical point at c_{i+1}. But the only branch point of ψ_{i+1}^{-1} which is an

A-point is the one at which the branches of φ_{i+1}^{-1} are permuted in a single cycle. Hence $c_{i+1} = c'_{i+1}$.

When $k = 2$, the value of φ_i^{-1} at its simple point at b_i is the affix of that critical point of ψ_{i+1}^{-1} at which the branches of φ_{i+1}^{-1} are permuted in pairs. We shall show that the value of φ_i^{-1} at its simple point at a_i is the affix of the second critical point of φ_{i+1}^{-1} where its branches are permuted in pairs.

As, by (11), $F = \psi_i \varphi_{i+1}$, we see that F^{-1} has precisely two uniform branches at a_i, whose values are the values of φ_{i+1}^{-1} at its simple points. One of these values is the affix of a critical point of φ_{i+2}^{-1} at which the branches of φ_{i+2}^{-1} are permuted in pairs. From $F = \varphi_i \psi_{i+1}$ we see now that at the point whose affix is the value of φ_i^{-1} at its simple point at a_i, both branches of ψ_{i+1}^{-1} are uniform, and the value of at least one of them is the affix of a critical point of φ_{i+2}^{-1}. This can be so only if the value of φ_i^{-1} at its simple point at a_i is the affix of a critical point of φ_{i+1}^{-1} where the branches of φ_{i+1}^{-1} are permuted in pairs.

From what precedes, we see that when $h = 3$, and $\alpha_i = \beta_i = 2$, both φ_i^{-1} and ψ_i^{-1} have two simple points at a_i and b_i, and the values of φ_i^{-1} and ψ_i^{-1} at their simple points are the same. We shall call these values a_{i+1} and b_{i+1}. Also, at c_i, φ_i^{-1} and ψ_i^{-1} both have their branches permuted in a single cycle, and assume a common single value, which we shall call c_{i+1}.

It is hardly necessary to mention the fact that for every j greater than the i used in our work above, φ_j^{-1} and ψ_j^{-1} will have the properties proved for φ_i^{-1} and ψ_i^{-1}.

Precisely as in the preceding section, we can show that σ_i^{-1} has no critical points other than a_{i+1}, b_{i+1} and c_{i+1}.

We shall prove now that all branch points which σ_i^{-1} has at a_{i+1} and b_{i+1} are simple.

Suppose that for some $j \geqq i$, some of the branch points of σ_j^{-1} at a_{j+1} and b_{j+1} are not simple, and let m_j be the sum of the orders of the branch points which are not simple. Since $m_j \leqq 2t-2$, there is a j for which m_j is a maximum. We deal with such a j.

Consider the relation

$$\sigma_j \varphi_j = \varphi_{j+1} \sigma_{j+1}. \tag{12}$$

We see that the inverse of $\sigma_j \varphi_j$ has at a_{j+1} and b_{j+1} branch points which are not simple, and the sum of whose orders is at least rm_j. As the critical points of σ_{j+1}^{-1} which are values of φ_{j+1}^{-1} at a_{j+1} and b_{j+1} are a_{j+2} and b_{j+2}, the values of φ_{j+1}^{-1} at its simple points, and as the branch points of φ_{j+1}^{-1} at a_{j+1} and b_{j+1} are all simple, we see that σ_{j+1}^{-1} has branch points at a_{j+2} and b_{j+2} which are not simple, and the sum of whose orders is at least rm_j. This contradicts the assumption that m_j is a maximum.

Thus, the inverse of $\varphi_i \sigma_i$ has no critical point other than a_i, b_i, c_i, and at a_i and b_i its branch points are all simple.

If every branch of σ_i^{-1} were permuted at a_{i+1} and at b_{i+1}, σ_i^{-1} would have just two distinct values at c_{i+1}, for its index at c_{i+1} would be $t-2$. Thus φ_i^{-1}, since it has three critical points, would have at least one critical point which is a value assumed by σ_i^{-1} at some point other than c_{i+1}. This means that the inverse of $\sigma_i \varphi_i$ would either have more critical points than a_{i+1}, b_{i+1}, c_{i+1}, or else it would have branch points of order greater than unity at a_{i+1} or b_{i+1}. This is impossible by (12), according to what we know of the critical points of the second member of (12). Hence there are, at a_{i+1} and b_{i+1}, precisely two places on the Riemann surface of σ_i^{-1} at which σ_i^{-1} is uniform. By (12), the value of σ_i^{-1} at these are a_i and b_i. Also, at c_{i+1}, the branches of σ_i^{-1} are permuted in a single cycle, and $\sigma_i^{-1}(c_{i+1}) = c_i$.

Without difficulty, we see now that the inverse of

$$F = \varphi_i \sigma_i \psi_i \sigma_i$$

has the two critical points a_i and b_i at which its branches are permuted in pairs, and the critical point c_i at which its branches are permuted in a single cycle. Also, at a_i and b_i, there are two places on the surface of F^{-1} at which F^{-1} is uniform, and the values of F^{-1} at these places are a_i and b_i. Finally, $F^{-1}(c_i) = c_i$.

From the relation

$$\varphi_i \sigma_i \psi_i \sigma_i = \sigma_{i-1} \varphi_{i-1} \sigma_{i-1} \psi_{i-1},$$

we see that σ_{i-1}^{-1} has no critical points other than a_i, b_i, c_i, that at a_i and b_i its branch points are all simple, and that at c_i its branches are permuted in a single cycle. Also, at a_i and b_i there are two places on the surface of σ_{i-1}^{-1} at which σ_{i-1}^{-1} is uniform, assuming certain values a_{i-1} and b_{i-1}.

We let $\sigma_{i-1}^{-1}(c_i) = c_{i-1}$. It is easy to see that the inverse of

$$\varphi_{i-1} \sigma_{i-1} \psi_{i-1}$$

has a critical point at c_{i-1} at which its branches are permuted in a single cycle. At a_{i-1} and b_{i-1} its branches are permuted in pairs, except that there are two places where the inverse is uniform and takes the values a_i and b_i. Then the inverse of

$$\varphi_{i-1} \sigma_{i-1} \psi_{i-1} \sigma_{i-1}$$

has all its branches permuted in a single cycle at c_{i-1}; and all its branches permuted in pairs at a_{i-1} and b_{i-1}, except that there are two places where the inverse is uniform and takes the values a_{i-1} and b_{i-1}. The value at c_{i-1} is c_{i-1}.

Continuing thus, we prove that the inverse of

$$\Phi \Psi = \Psi \Phi$$

has three critical points a_0, b_0, c_0, where it behaves in the manner already frequently described.

Also, Φ^{-1} and Ψ^{-1} have the critical points a_0, b_0, c_0. It is obvious that $\Phi^{-1}(c_0) = \Psi^{-1}(c_0) = c_0$. Furthermore, as Φ and Ψ are at least of degree 4, their inverses both actually have critical points at a_0 and b_0, so that the values which each inverse assumes at those places at a_0 and b_0 where it is uniform are a_0 and b_0.

When the sequence (C) does not exist, we may take $i = 0$, so that $\varphi_0 = \varphi_1 = \Phi$, $\psi_0 = \psi_1 = \Psi$. The values a_1, b_1, c_1, which Φ^{-1} and Ψ^{-1} assume at their simple points and at c_0, are seen directly to be the same as a_0, b_0, c_0.

Let $\lambda(z)$ be a linear function such that

$$\lambda(a_0) = 1, \qquad \lambda(b_0) = -1, \qquad \lambda(c_0) = \infty.$$

Then the inverses of the two permutable functions

$$\Phi_1 = \lambda \Phi \lambda^{-1}, \qquad \Psi_1 = \lambda \Psi \lambda^{-1}$$

have simple branch points at 1 and -1, and their branches are permuted in a single cycle at ∞. The values of their uniform branches at 1 and -1 are 1 and -1, and $\Phi_1^{-1}(\infty) = \Psi_1^{-1}(\infty) = \infty$.

Consider the function $\cos z$. Wherever it assumes either of the values 1 or -1, it assumes it twice. It never assumes the value ∞. Hence, if we operate on $\cos z$ with Φ_1^{-1}, the n values of $\Phi_1^{-1}(\cos z)$ are uniform *im kleinen*, and therefore, also, uniform *im großen*. As Φ_1^{-1} assumes the value ∞ only at ∞, these n functions are entire.

Let $f(z)$ be one of these entire functions. As $\Phi^{-1}(\cos z)$ assumes a value 1 or -1 only when $\cos z$ is 1 or -1, and then only through a uniform branch of Φ_1^{-1}, $f(z)$ cannot assume a value 1 or -1 unless it assumes it twice. Also, $f(z)$ is never infinite.

Consider the function arc cos z. Its only finite critical points are 1 and -1, at which its branches are permuted in pairs. Hence, by the same reasoning used above, there are an infinite number of entire functions arc cos $f(z)$. Let $\beta(z)$ be one of these. We shall show that $\beta(z)$ is linear.

Wherever $\Phi_1^{-1}(z)$ is large, its modulus is of the order of $\sqrt[m]{|z|}$, where m is the degree of Φ_1. Now, since

$$|\cos z| \leqq e^{|z|},$$

there is a k such that

$$|f(z)| < k\, e^{|z/m|},$$

for every z.

Now if

$$\beta(z) = u(x, y) + iv(x, y)$$

were not linear, there would be values of z of large modulus for which

$$v(x, y) < -|z|,$$

and hence for which

$$|f(z)| = |\cos \beta(z)| \geqq \frac{e^{|z|} - e^{-|z|}}{2}.$$

This contradicts the first inequality for $|f(z)|$.

Thus there is a relation

$$\cos z = \Phi_1[\cos(pz + q)],$$

or, what is the same, a relation

$$\cos(az + b) = \Phi_1(\cos z). \tag{13}$$

Similarly, we have

$$\cos(cz + d) = \Psi_1(\cos z).$$

As the first member of (13) has the primitive period $2\pi/a$, and as the second member has a period 2π, we see that a is an integer. In fact, we must have $a = \pm m$, where m is the degree of Φ_1. To determine b, we note that the first member of (13) must be, like the second, an even function of z. Hence, when

$$z_1 + z_2 = 2\pi,$$

we must have

$$az_1 + az_2 + 2b = 2k\pi.$$

where k is some integer. It follows that $2b$ is a multiple of 2π, so that b is either 0 or π (neglecting multiples of 2π). Similarly, $c = \pm n$, and d is either 0 or π.

Finally, since

$$\cos(acz + ad + b) = \Phi_1 \Psi_1(\cos z),$$

$$\cos(acz + cb + d) = \Psi_1 \Phi_1(\cos z),$$

we must have

$$(a-1)d \equiv (c-1)b \qquad (\mathrm{mod}\, 2\pi).$$

As in the preceding section, all permutable pairs of functions of the type now considered can be found by transforming $\Phi_1(z)$ and $\Psi_1(z)$ with a linear function.

VI. The multiplication formulas for $\wp z$

We are going to settle, in this section, the following two cases:

(a) $h = 4$,

(b) $h = k = 2$ and $r = s = 2$. (See next to last paragraph of § III.)

Let us examine Case (a). We must have $k = 4$, 3 or 2.

Suppose that $k = 4$. Then, by § III, φ_i^{-1} and ψ_i^{-1} must have the same four critical points, at which each has, in all, four simple points. Furthermore, the four values which φ_i^{-1} assumes at its simple points are the same that ψ_i^{-1} assumes at its simple points, for these four values are the affixes of the common critical points of φ_{i+1}^{-1} and ψ_{i+1}^{-1}.

If $k = 3$, the degree s of ψ_i^{-1} must be 4, and ψ_i^{-1} must have three critical points at which its branches are permuted in pairs. The four simple points of ψ_i^{-1} are at a critical point w_0 of φ_i^{-1}. As the four critical points of φ_{i+1}^{-1} will all have the same index that φ_i^{-1} has at w_0, that index must be $(r-1)/2$. Hence r is odd, and φ_i^{-1} has one simple point at each of its four critical points.

Suppose that $k = 2$. We must have $s = 2$. Two of the critical points of φ_i^{-1} are critical points of ψ_i^{-1}. The other two, call them w_1 and w_2, are not.

We shall show that the four values which φ_i^{-1} assumes at its simple points are the affixes of the four critical points of φ_{i+1}^{-1}, that is, the four values of ψ_i^{-1} at w_1 and w_2.

Since φ_{i+1}^{-1} has two critical points of the type that φ_i^{-1} has at w_1, and two of the type that φ_i^{-1} has at w_1, it follows that φ_i^{-1} has (in all) two simple points at w_1 and w_2. The other two simple points of φ_i^{-1} are A-points of φ_i^{-1}, and the values which φ_i^{-1} takes at them are critical points of φ_{i+1}^{-1}. We write

$$F = \varphi_i \, \psi_{i+1} = \psi_i \, \varphi_{i+1}.$$

From the relation $F = \psi_i \, \varphi_{i+1}$, we see that, at w_1 and w_2, there are precisely four places on the Riemann surface of F^{-1} at which F^{-1} is uniform, and that, at these four places, the values of F^{-1} are the values which φ_{i+1}^{-1} assumes at its four simple points. Let u_1 and u_2 be the values of φ_i^{-1} at its simple points at w_1 and w_2. From what we have just seen, and from the relation $F = \varphi_i \, \psi_{i+1}$, it follows that neither u_1 nor u_2 is a critical point of ψ_{i+1}^{-1}, and that the values of ψ_{i+1}^{-1} at u_1 and u_2 are the four values of φ_{i+1}^{-1} at its simple points. As at least two of these values are critical points of φ_{i+2}^{-1}, it follows that at least one of the points u_1 and u_2 is a critical point of φ_{i+1}^{-1}. We have proved that at least three of the values of φ_i^{-1} at its simple points are critical points of φ_{i+1}^{-1}. This implies, of course, that at least three of the values of φ_{i+1}^{-1} at its simple points are critical points of φ_{i+2}^{-1}. Going back three sentences, we see that u_1 and u_2 are both critical points of φ_{i+1}^{-1}, as was to be proved.

By similar reasoning, only more briefly, it can be shown that, when $k = 3$, φ_i^{-1} and ψ_i^{-1} assume the same four values at their simple points.

We shall now examine Case (b), and show that it may be amalgamated with Case (a).

Let a_i and b_i be the critical points of φ_i^{-1}, and a_i and c_i those of ψ_i^{-1}. The inverse of

$$F = \varphi_i \, \psi_{i+1} = \psi_i \, \varphi_{i+1}$$

has the three critical points a_i, b_i, c_i, at which its branches are permuted in pairs.

As φ_{i+1}^{-1} and ψ_{i+1}^{-1} have precisely one critical point in common, one, and only one, of the values of φ_i^{-1} at c_i equals a value of ψ_i^{-1} at b_i. Let the values of φ_i^{-1} at c_i be a_{i+1} and c_{i+1}, and the values of ψ_i^{-1} at b_i be a_{i+1} and b_{i+1}. There is a point d_i at which φ_i^{-1} has the value b_{i+1}. Evidently d_i is distinct from c_i. We shall show that d_i is distinct from a_i and from b_i.

As a_{i+1} and b_{i+1} are the critical points of φ_{i+1}^{-1}, and a_{i+1} and c_{i+1} those of ψ_{i+1}^{-1}, one and only one value of φ_{i+1}^{-1} at c_{i+1} equals a value of ψ_{i+1}^{-1} at b_{i+1}. Let the values of φ_{i+1}^{-1} at c_{i+1} be a_{i+2} and c_{i+2}, and those of ψ_{i+1}^{-1} at b_{i+1} be a_{i+2} and b_{i+2}. From $F = \varphi_i \, \psi_{i+1}$, we see that where φ_i^{-1} takes the value b_{i+1}, F^{-1} has branches with values a_{i+2} and b_{i+2}. But from $F = \psi_i \, \varphi_{i+1}$, we see that at the same point at which one branch of F^{-1} has the value a_{i+2}, another has the value c_{i+2}. Hence where F^{-1} takes the value a_{i+2}, it has at least three values, and therefore four. Thus F^{-1} cannot assume the value a_{i+2} at either a_i or b_i, as it has only two distinct values at these points. This proves that d_i is distinct from a_i and c_i.

Let b_{i+1} and d_{i+1} be the two values of φ_i^{-1} at d_i. We shall prove that the values of ψ_i^{-1} at d_i are c_{i+1} and d_{i+1}.

The relation $F = \varphi_i \psi_{i+1}$ shows that two of the values of F^{-1} at d_i are a_{i+2} and b_{i+2}. It follows from $F = \psi_i \varphi_{i+1}$ that ψ_i^{-1} assumes the value c_{i+1} at d_i.

Thus, since $F = \psi_i \varphi_{i+1}$, three of the values of F^{-1} at d_i are $a_{i+2}, b_{i+2}, c_{i+2}$. Hence, from $F = \varphi_i \psi_{i+1}$, we see that ψ_{i+1}^{-1} assumes the value c_{i+2} at d_{i+1}. But the proof above that φ_i^{-1} assumes the value b_{i+1} at the same point at which ψ_i^{-1} assumes the value c_{i+1} proves also that if ψ_{i+1}^{-1} assumes the value c_{i+2} at d_{i+1}, φ_{i+1}^{-1} must assume the value b_{i+2} at d_{i+1}. Hence, from $F = \psi_i \varphi_{i+1}$, we find that ψ_i^{-1} assumes the value d_{i+1} at d_i.

Returning to Case (a), we denote the critical points of φ_i^{-1} and ψ_i^{-1}, for every $i \geqq i_1$, by a_i, b_i, c_i, d_i. This will permit us to treat Cases (a) and (b) together.

The functions σ are introduced as in the two preceding cases. It is seen immediately that σ_i^{-1} has no critical points other than $a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$, and that its branch points are all simple. This shows that the inverses of $\varphi_i \sigma_i$ and $\psi_i \sigma_i$ have no critical points other than a_i, etc., that their branch points are all simple, and that there are exactly four places on the Riemann surfaces of the inverses at a_i, etc., where the inverses are uniform.

When $h = 4$, the relation

$$\sigma_i \varphi_i = \varphi_{i+1} \sigma_{i+1} \tag{14}$$

shows that the values which σ_i^{-1} assumes where it is uniform at a_{i+1}, etc., are a_i etc. We shall show that the same holds when $h = 2$. First, the above relation shows that two of these values of σ_i^{-1} are a_i and b_i. Similarly, the relation $\sigma_i \psi_i = \psi_{i+1} \sigma_{i+1}$ shows that two of the values are a_i and c_i. It remains to show that the fourth value is d_i. If it were not, then since the values of φ_i^{-1} at d_i are b_{i+1} and d_{i+1}, the inverse of $\sigma_i \varphi_i$ would take the values b_{i+1} and d_{i+1} at some point distinct from a_i, etc. But the argument just given for σ_i^{-1} shows that σ_{i+1}^{-1} assumes the values $a_{i+1}, b_{i+1}, c_{i+1}$, at a_{i+2} etc., which means that the inverse of $\varphi_{i+1} \sigma_{i+1}$ assumes those three values at a_{i+1} etc. This, with (14), yields a contradiction.

It is visible now that the inverse of

$$F = \varphi_i \sigma_i \psi_i \sigma_i$$

has the four critical points a_i, etc., that its branch points are all simple, that there are just four places on the surface of F^{-1} at a_i, etc., where F^{-1} is uniform, and that the values of F^{-1} at these places are a_i, etc.

We now work back to Φ and Ψ. From the relation

$$(15) \qquad F = \varphi_i \sigma_i \psi_i \sigma_i = \sigma_{i-1} \varphi_{i-1} \sigma_{i-1} \psi_{i-1},$$

we see that σ_{i-1}^{-1} has no critical points other than a_i, etc., that its branch points are all simple, and that there are, at a_i, etc., just four places on the surface of σ_{i-1}^{-1} at which σ_{i-1}^{-1} is uniform. Let the values of σ_{i-1}^{-1} at these four places be a_{i-1}, etc. We see by (15) that the inverse of $\varphi_{i-1} \sigma_{i-1} \psi_{i-1}$ has the critical points a_{i-1}, etc. (all four, because it has at least eight branches), where its branches are permuted in pairs, except that there are four places at a_{i-1}, etc., where the inverse is uniform and takes the values a_i, etc. Then the inverse of

$$\varphi_{i-1} \sigma_{i-1} \psi_{i-1} \sigma_{i-1}$$

has the four critical points a_{i-1}, etc., with simple branch points, and has four places at a_{i-1}, etc., where it is uniform and takes the values a_{i-1}, etc.

Proceeding thus, we find that the inverse of

$$(16) \qquad \Phi \Psi = \Psi \Phi$$

has four critical points a_0, b_0, c_0, d_0, at which its branches are permuted in pairs, except that there are four places where the inverse is uniform, assuming the values a_0, etc.

As to Φ^{-1} and Ψ^{-1}, we see that they have no critical points other than a_0, etc., and that their branch points are all simple. When Φ^{-1} and Ψ^{-1} each have four critical points, it is evident that their values where they are uniform at a_0, etc., are a_0, etc.

If $m > 4$, Φ^{-1} will have four critical points, so that the values of Ψ where it is uniform at a_0, etc., are a_0, etc. Suppose that $m > 4$, that $n = 4$, and that Ψ^{-1} has only three critical points. It is clear that at three of the places at a_0, etc., at which Φ^{-1} is uniform, it assumes values from among a_0, etc. If it did not assume one of these values at the fourth place, we would have the contradiction that the first member of (16) has four uniform branches at one of the points a_0, etc., which do not assume the values a_0, etc.

When $m = 4$, we must also have $n = 4$, so that φ_i, ψ_i, σ_i, are all of degree 2, and we may assume, above, that $i = 0$. It was shown above that in this case the values of the inverses of $\varphi_i \sigma_i$ and $\psi_i \sigma_i$, where they are uniform at a_i, etc., are a_i etc.

Suppose that the sequence (C) does not exist. We may take $i = 0$. As a_1, etc., play the same rôle in regard to φ_1 and ψ_1 as a_0, etc., do in regard to φ_0 and ψ_0, and as $\varphi_0 = \varphi_1 = \Phi$, $\psi_0 = \psi_1 = \Psi$, we see immediately that the values which Φ^{-1} and Ψ^{-1} take at their uniform places at a_0, etc., are a_0, etc.

Let $\lambda(z)$ be a linear function such that $\lambda(a_0) = \infty$ and that

$$\lambda(b_0) + \lambda(c_0) + \lambda(d_0) = 0.$$

It does not matter, in this, which point is called a_0. We put

$$e_1 = \lambda(b_0), \qquad e_2 = \lambda(c_0), \qquad e_3 = \lambda(d_0).$$

We consider the two functions

$$\Phi_1 = \lambda \Phi \lambda^{-1}, \qquad \Psi_1 \quad \lambda \Psi \lambda^{-1},$$

whose inverses have only simple branch points, which are found at e_1, e_2, e_3, ∞.

Construct the elliptic function $\wp z$ such that $\wp(\omega_i) = e_i$, $i = 1, 2, 3$. This is possible because $e_1 + e_2 + e_3 = 0$. Furthermore, the orientation of the numbers e_i is of no importance.

As $\wp z$ assumes the values e_i and ∞ twice wherever it assumes one of them, and as the branch points of Φ_1^{-1} are all simple, the n values of $\Phi_1^{-1}(\wp z)$ are uniform *im kleinen*, and hence also *im großen*. Let $f(z)$ represent one of the n meromorphic functions $\Phi_1^{-1}(\wp z)$. As Φ_1^{-1} assumes a value e_i or ∞ only where it is uniform, $f(z)$ cannot assume a value e_i or ∞ unless it assumes it twice.

Let $\wp^{-1} z$ be the inverse of $\wp z$. Then $\wp^{-1} z$ has the four critical points e_i and ∞, and its branch points are all simple. Hence there are an infinity of meromorphic functions $\wp^{-1}[f(z)]$. Let $\beta(z)$ be one of these. Then

$$f(z) = \wp[\beta(z)],$$

or

(17) $$\wp z = \Phi_1 \wp [\beta(z)].$$

Now $\wp[\beta(z)]$, being algebraically related to $\wp z$, is an elliptic function. Then $\beta(z)$ must be entire, for if it had a pole with a finite affix $\wp[\beta(z)]$ would have an essential singularity with a finite affix.

Differentiating (17), we find

$$\wp'(z) = \Phi_1' \wp[\beta(z)] \wp'[\beta(z)] \beta'(z). \tag{18}$$

Now $\wp'[\beta(z)]$, being algebraically related to $\wp[\beta(z)]$, is an elliptic function. It follows from (18) that $\beta'(z)$ is an elliptic function. As $\beta'(z)$ is entire, it must be a constant. Thus $\beta(z)$ is linear.

There exists thus a relation

$$\wp(z) = \Phi_1[\wp(pz+q)],$$

or, what is the same, a relation

$$\wp(az+b) = \Phi_1(\wp z). \tag{19}$$

Similarly, there is a relation

$$\wp(cz+d) = \Psi_1(\wp z). \tag{20}$$

It remains only to characterize the constants. From (19) we see first, since $2\omega_1$ and $2\omega_3$ are periods of the first member, that

$$2a\omega_1 \equiv 0, \qquad 2a\omega_3 \equiv 0 \qquad (\text{modd}\, 2\omega_1, 2\omega_3). \tag{21}$$

Also, since the first member of (19) has to be an even function of z, like the second, we find

$$2b \equiv 0 \qquad (\text{modd}\, 2\omega_1, 2\omega_3). \tag{22}$$

From (20), we find, similarly,

$$2c\omega_1 \equiv 0, \qquad 2c\omega_3 \equiv 0, \qquad 2b \equiv 0 \qquad (\text{modd}\, 2\omega_1, 2\omega_3). \tag{23}$$

Finally, from the condition of permutability, we find

$$(a-1)d \equiv (c-1)b \qquad (\text{modd}\, 2\omega_1, 2\omega_3). \tag{24}$$

Any set of constants which satisfy (21), (22), (23), (24) yield a pair of permutable functions. All pairs of permutable functions, of the type considered in this section, are found by transforming Φ_1 and Ψ_1 with some linear function $\lambda(z)$.

VII. THE MULTIPLICATION FORMULAS FOR $\wp^2 z$ IN THE LEMNISCATIC CASE

We consider here the case in which $h = 3$, and in which, for some $i \geqq i_1$, one of the numbers α_i, β_i, γ_i is 2, and the other two 4. Let, for instance, $\alpha_i = 2$, and $\beta_i = \gamma_i = 4$.

We take first the case of $k = 3$, in which φ_i^{-1} and ψ_i^{-1} have the common critical points a_i, b_i, c_i. As before, we represent the orders of the substitutions which the branches of ψ_i^{-1} undergo at these points by α_i', β_i', γ_i', respectively.

We shall prove that $\alpha_i' = 2$, and that, when $s > 4$, $\beta_i' = \gamma_i' = 4$. When $s = 4$, one of β_i' and γ_i' is 2, and the other is 4. We cannot have $s = 3$.

First let r be odd. Since α_i, β_i, γ_i are divisors of 4, φ_i^{-1} must have at least one simple point at each of a_i, b_i, c_i. If α_i' were not 2, or if β_i' and γ_i' were not divisors of 4, φ_i^{-1} would have A-points in addition to the three simple points.

Suppose that one of β_i' and γ_i' is 2 rather than 4. Let it be γ_i', for instance. Unless $s \leqq 6$. ψ_i^{-1} will surely have more than three A-points at c_i. If $s = 6$, ψ_i^{-1} must have three simple branch points c_i. This leads to the result that φ_{i+1}^{-1} has three critical points at which its branches are permuted in pairs, a situation proved impossible in § V. If $s = 5$, ψ_i^{-1} would have at least three A-points at c_i, and at least one at a_i. If $s = 3$, ψ_i^{-1} would have at least two A-points at c_i, and at least one at each of a_i and b_i.

Thus, when $\gamma_i' = 2$, we have $s = 4$. We must have $\beta_i' = 4$, for the case of $\alpha_i' = \beta_i' = \gamma_i' = 2$ is known to be impossible.

When $s = 4$, it is necessary, since $\beta_i' = 4$, that ψ_i^{-1} have two simple points, either at a_i, or at c_i. They must be at c_i, otherwise ψ_i^{-1} would have four A-points.

We take now the case in which r is even. It is evident that $r > 4$, so that r is at least 6.

First we prove that $\alpha_i' = 2$. If $\alpha_i' > 2$, r must be 6, and φ_i^{-1} must have three simple branch points at a_i. As φ_i^{-1} has branch points of order 3 at b_i and at c_i, φ_i^{-1} must have two simple points at b_i or at c_i, or the sum of its indices would be 11 instead of 10. Thus, φ_i^{-1} would have at least five A-points. Hence $\alpha_i' = 2$.

As $r \geqq 6$, φ_i^{-1} has at least four points at b_i and c_i. Then either β_i' or γ_i' must be a divisor of 4. Let it be β_i', for instance.

Let $\beta_i' = 2$. The very proof used above for r odd shows that $s = 4$. We know that γ_i' cannot equal 2 in this case, and it will be seen below that $\gamma_i' \neq 3$. Hence, if it were possible for s to be 4, we would have $\gamma_i' = 4$. This information will be used below in proving that either r or s must be odd; we shall know thus that β_i' cannot be 2.

Suppose that $s > 4$, so that $\beta'_i = 4$. By (9), we must have $\gamma'_i = 3$ or 4. We show that γ'_i cannot be 3 for any value of s. If γ'_i were 3, at least two of the orders α'_{i+1}, β'_{i+1}, γ'_{i+1} of the substitutions at the critical points of ψ_{i+1}^{-1} must equal 3. The orders α_{i+1}, β_{i+1}, γ_{i+1}, at the critical points of φ_{i+1}^{-1}, are certainly divisors of 4. They cannot all be 4. If two of them are 2, we see from § V that two of α'_{i+1}, etc. must be 2. If one of α_{i+1}, etc. is 2 and the other two 4, the argument above shows that two of α'_{i+1}, etc. are divisors of 4. Hence $\gamma'_i = 4$.

We shall now examine the Riemann surfaces of φ_i^{-1} and ψ_i^{-1}. It will suffice to deal with φ_i^{-1}.

Suppose first that r is odd. Then the indices of φ_i^{-1} at a_i, b_i, c_i are not greater, respectively, than

$$\frac{r-1}{2}, \qquad \frac{3(r-1)}{4}, \qquad \frac{3(r-1)}{4}.$$

As the sum of the three indices is $2r-2$, the upper bounds just given must be the actual values of the indices. Thus φ_i^{-1} has one simple point and $(r-1)/2$ simple branch points at a_i, and one simple point and $(r-1)/4$ branch points of order 3 at b_i and at c_i.

Let r now be even. We shall prove that φ_i^{-1} has two simple points, either at b_i or at c_i. We know that α'_{i+1}, β'_{i+1}, γ'_{i+1} are divisors of 4. They cannot all be 2, as we have seen several times. Thus one of them at least must be 4. This is possible only if φ_i^{-1} has either at b_i or at c_i a point whose order is prime to 4. Such a point has to be a simple point if β_i and γ_i equal 4. Also, since r is even, the simple points of φ_i^{-1} must come in pairs.

Suppose that the two simple points are at c_i. If $r \equiv 2$, mod 4, there must be a simple branch point at b_i. If $r \equiv 0$, mod 4, there must be a simple branch point at c_i, in addition to the two simple points. The two simple points and the simple branch point are the A-points of φ_i^{-1}.

We show now that either r or s is odd. Suppose that both are even. By what precedes, we may suppose that ψ_i^{-1} has two simple points at c_i. Thus φ_{i+1}^{-1} would have two critical points with similar substitutions of order 4, whereas the preceding paragraph shows that the substitutions cannot be similar when r is even.

We consider now the case of $k = 2$. We must have $s = 3$, or $s = 2$. If s were 3, ψ_i^{-1} would have an A-point at each of its critical points, which would be points of order 3, and three A-points at that critical point of φ_i^{-1} which is not a critical point of ψ_i^{-1}. Hence, $s = 2$. If the critical points of ψ_i^{-1} were b_i and c_i, they would be A-points of ψ_i^{-1} and ψ_i^{-1} would also have two A-points at a_i. Thus we may assume that the critical points of ψ_i^{-1} are a_i and c_i.

We shall show that, when $k = 2$, r is odd. Suppose that r is even. If ψ_{i+1}^{-1} is to have critical points, φ_i^{-1} must have two simple points, either at a_i or at c_i. Suppose that they are at a_i. Consider the relation

$$F = \varphi_i \psi_{i+1} = \psi_i \varphi_{i+1}.$$

It follows from $F = \varphi_i \psi_{i+1}$ that among the values of F^{-1} at a_i are the two values which ψ_{i+1}^{-1} assumes as its critical points. One of these values is the affix of a critical point of φ_{i+2}^{-1} with a substitution of order 2, and hence the affix of a critical point of ψ_{i+2}^{-1}. Now, from $F = \psi_i \varphi_{i+1}$, since the affixes of the critical points of ψ_{i+2}^{-1} are values of φ_{i+1}^{-1} at its simple points, it follows that the value of ψ_i^{-1} at a_i is the affix of a critical point of φ_{i+1}^{-1}, a falsity.

If the two simple points are at c_i, φ_{i+1}^{-1} will have two simple points at its critical point with substitution of order 2, something just seen to be impossible.

Thus, when $k = 2$, r is odd, so that φ_i^{-1} has one simple point at each of a_i, b_i, c_i. Furthermore, it is easy to show, by the method already frequently used, that the value a_{i+1} which φ_i^{-1} assumes at its simple point at a_i is the value of ψ_i^{-1} at c_i, and that the values b_{i+1} and c_{i+1} which φ_i^{-1} assumes at its simple points at b_i and c_i are the two values of ψ_i^{-1} at b_i.

Suppose that r is odd. If s is odd, φ_i^{-1} and ψ_i^{-1} have the same value, a_{i+1}, at their simple points at a_i, and the same pair of values b_{i+1} and c_{i+1} at their simple points at b_i and c_i. If s is even, ψ_i^{-1} takes the values b_{i+1} and c_{i+1} at its simple points, and the value a_{i+1} at its simple branch point which is an A-point.

In what remains to be done, it is unnecessary to use the condition that $r \geqq s$. Accordingly, we work under the legitimate and convenient assumption that r is odd.

The details from this point on are entirely analogous to the corresponding details in the three cases already treated. It is the easiest matter to prove that σ_i^{-1} has no critical points other than a_{i+1}, b_{i+1}, c_{i+1}, and that the orders of the substitutions which its branches undergo at those points are divisors of 2, 4 and 4 respectively. Also we work back readily to Φ^{-1} and Ψ^{-1}. These have no critical points other than three certain points a_0, b_0, c_0, at which their branches undergo substitutions whose orders are divisors of 2, 4 and 4 respectively. If the degree of either permutable function is odd, the branches of its inverse are permuted in pairs at a_0, except one which is uniform and has the value a_0. At b_0 and c_0 its branches are permuted in fours, except that, both at b_0 and at c_0, there is a place on the surface of the inverse where the inverse is uniform, and the values of the inverse at these places are (as a pair) b_0 and c_0. If the degree is even, the branches are permuted in pairs at a_0. Either at b_0 or at c_0, there are two uniform branches whose values are b_0 and c_0, and a simple branch point, where the value assumed is c_0.

To identify Φ and Ψ, we choose any number ω, different from zero, and construct $\wp(z\,|\,\omega, i\omega)$. It is well known that in this lemniscatic case, $e_2 = 0$, and $e_3 = -e_1$. We consider now $\wp^2 z$. Where it assumes the value ∞, namely, at the points congruent to the origin, it assumes it four times. Similarly, the value 0 is assumed only at the points congruent to ω_2, and then four times, while the value e_1^2 is assumed twice at all points congruent to ω_1 and ω_3. There are no values other than ∞, 0, and e_1^2 which are assumed more than once by $\wp^2 z$ at any point.

We now take a linear function $\lambda(z)$ such that

$$\lambda(a_0) = e_1^2, \qquad \lambda(b_0) = 0, \qquad \lambda(c_0) = \infty,$$

and deal with

$$\Phi_1 = \lambda\,\Phi\,\lambda^{-1}, \qquad \Psi_1 = \lambda\,\Psi\,\lambda^{-1}.$$

Precisely as in the preceding section, we find that

$$\wp^2(az+b) = \Phi_1(\wp^2 z), \qquad \wp^2(cz+d) = \Psi_1(\wp^2 z),$$

where

$$2a\,\omega_i \equiv 0, \quad 2c\,\omega_i \equiv 0 \quad (\text{modd}\; 2\,\omega_1,\, 2\,\omega_3) \quad (i = 1, 3).$$

To determine b, we note that, in the lemniscatic case, $\wp^2 iz = \wp^2 z$. Hence, if

$$z_1 \equiv iz_2 \qquad (\text{modd}\; 2\,\omega_1,\, 2\,\omega_3),$$

we must have,

$$az_1 + b \equiv iaz_2 + ib \qquad (\text{modd}\; 2\,\omega_1,\, 2\,\omega_3).$$

Multiplying the first congruence through by a, and subtracting the result from the second, we have

$$b(1-i) \equiv 0 \qquad (\text{modd } 2\omega_1, 2\omega_3).$$

A similar condition holds for d. Furthermore, as in the preceding cases

$$(a-1)d \equiv (c-1)b \qquad (\text{modd } 2\omega_1, 2\omega_3).$$

VIII. The multiplication formulas for $\wp' z$ in the equianharmonic case

We take, for $h = 3$, the case in which, for some $i \geqq i_1$, $\alpha_i = \beta_i = \gamma_i = 3$.

We cannot have $r \equiv 2$, mod 3, for in that case, the sum of the indices of φ_i^{-1} could be at most $2r-4$.

If $r \equiv 1$, mod 3, there is one simple point at each critical point of φ_i^{-1}, and the other points are all of order 3.

If $r \equiv 0$, mod 3, there are three simple points at one of the critical points, and none at either of the others.

Let $k = 3$. Then $\alpha_i' = \beta_i' = \gamma_i' = 3$, else φ_i^{-1} would have more A-points than its three simple points. The surface of ψ_i^{-1} is of one of the types described above.

If $k = 2$, we must have $s = 3$, for if s were 2, ψ_i^{-1} would have four A-points. When $k = 2$, we must have $r \equiv 1$, mod 3, and it can be shown, as in the preceding sections, that φ_i^{-1} and ψ_i^{-1} assume the same three values at their simple points.

It is most easy now to introduce σ_i and to work back to Φ and Ψ. The inverses of these two functions have no critical points other than certain three points a_0, b_0, c_0. On the Riemann surface of the inverse of either function, there are precisely three places at a_0, b_0, c_0, at which the inverse is uniform, and the values of the inverse at these places are a_0, b_0, c_0.

To identify Φ and Ψ, we construct $\wp(z \mid \omega, e^{\pi i/8}\omega)$ where ω is any number different from zero. As $\wp(z \mid \omega_1, \omega_3)$ is a homogeneous function of degree -2 in z, ω_1 and ω_3, and as $\wp(z \mid \omega, e^{\pi i/8}\omega)$ is identical with $\wp(z \mid e^{2\pi i/8}\omega, -\omega)$, we have, in the present case,

$$\wp e^{\frac{2\pi i}{8}} z = e^{\frac{2\pi i}{8}} \wp z.$$

Differentiating, we find

$$\wp' e^{\frac{2\pi i}{3}} z = \wp' z, \qquad \wp'' e^{\frac{2\pi i}{3}} z = e^{\frac{4\pi i}{3}} \wp'' z,$$

$$\wp''' e^{\frac{2\pi i}{3}} z = e^{\frac{2\pi i}{3}} \wp''' z.$$

Hence, if

(25) $$e^{\frac{2\pi i}{3}} z \equiv z \qquad (\text{modd } 2\omega,\ 2e^{\frac{\pi i}{3}}\omega),$$

we will have, except for $z \equiv 0$, when $\wp' z = \infty$,

(26) $$\wp'' z = \wp''' z = 0.$$

We consider the following two solutions of (25):

$$z_1 = \frac{2\omega}{1 - e^{\frac{2\pi i}{3}}}, \qquad z_2 = \frac{2\omega + 2e^{\frac{\pi i}{3}}\omega}{1 - e^{\frac{2\pi i}{3}}}.$$

We cannot have $z_1 \equiv z_2$, modd 2ω, $2e^{\frac{\pi i}{3}}\omega$, for

$$z_2 - z_1 \equiv 2\omega \frac{e^{\frac{\pi i}{3}}}{1 - e^{\frac{2\pi i}{3}}},$$

and the modulus of the second member of the last congruence is less than that of 2ω, the period of smallest modulus. We may suppose that z_1 and z_2 lie within the same parallelogram.

From (26), it follows that the values $\wp'(z_1)$ and $\wp'(z_2)$ are each assumed at least three times by $\wp'(z)$ at z_1 and z_2. But since $\wp''(z)$ vanishes just four times in a parallelogram, these values are assumed precisely three times each. Furthermore, no other value, except ∞, is ever assumed more than once at a point. Also, as $\wp'(z)$ assumes every value just three times in a parallelogram, $\wp'(z_1)$ and $\wp'(z_2)$ are not equal.

We now take a linear $\lambda(z)$ such that

$$\lambda(a_0) = \infty, \qquad \lambda(b_0) = \wp'(z_1), \qquad \lambda(c_0) = \wp'(z_2),$$

and let

$$\Phi_1 = \lambda \varphi \lambda^{-1}, \qquad \Psi_1 = \lambda \psi \lambda^{-1}.$$

We find

$$\wp'(az+b) = \Phi_1(\wp' z), \qquad \wp'(cz+d) = \Psi_1(\wp' z),$$

where

$$2a\omega \equiv 0, \qquad 2c\omega \equiv 0 \qquad (\text{modd}\, 2\omega,\, 2e^{\frac{\pi i}{3}}\omega).$$

$$\left(1 - e^{\frac{2\pi i}{3}}\right) b \equiv 0, \quad \left(1 - e^{\frac{2\pi i}{3}}\right) d \equiv 0 \quad (\text{modd}\, 2\omega,\, 2e^{\frac{\pi i}{3}}\omega),$$

and

$$(a-1)d \equiv (c-1)b \qquad (\text{modd}\, 2\omega,\, 2e^{\frac{\pi i}{3}}\omega).$$

IX. The multiplication formulas for $\wp^3 z$ in the equianharmonic case

Referring to (9), we find that the only case still to be treated is that, under $h = 3$, in which, for some $i \geqq i_1$, one and only one of α_i, β_i, γ_i equals 2, and a second equals 3. We let $\alpha_i = 2$, $\beta_i = 3$, $\gamma_i \neq 2$. We may assume that $\alpha_j = 2$, $\beta_j = 3$, $\gamma_j \neq 2$ for every $j > i$, for the work of the preceding sections shows that a different set of values of α_j, etc., would imply other values than those assumed for α_i, etc.

We prove first that $k = 3$. If k were 2, we would have $s = 3$ or $s = 2$. If s were 3, ψ_i^{-1} would have three simple points at that critical point of φ_i^{-1} which is not a critical point of ψ_i^{-1}. This leads to the contradiction that $\alpha_{i+1} = \beta_{i+1} = \gamma_{i+1}$. Suppose that $s = 2$. Then a_i must be a critical point of ψ_i^{-1}, for if the critical points of ψ_i^{-1} were b_i and c_i, ψ_i^{-1} would have an A-point at each of these two points, in addition to two A-points at a_i. The only way in which α_{i+1} can equal 2 is for c_i to be the second critical point of ψ_i^{-1}, and for γ_i to equal 4. Then $\alpha_{i+1} = 2$, $\beta_{i+1} = \gamma_{i+1} = 3$, which is impossible, for the argument just presented shows that one of β_{i+1} and γ_{i+1} must be 4, as γ_i is. Thus $k = 3$.

We must have, of course, $3 \leqq \gamma_i \leqq 6$. We shall show that $\gamma_i = 3$ when $r = 4$, and that otherwise $\gamma_i = 6$.

Suppose first that $\gamma_i = 3$. We cannot have $r \equiv 2$, mod 3, for φ_i^{-1} would have four simple points at b_i and c_i, which would be A-points. Suppose that $r \equiv 0$, mod 3. If there are no simple points at b_i or at c_i, the sum of the indices of φ_i^{-1} at b_i and c_i is $4r/3$. Hence the index at a_i is $2r/3 - 2$. This means, since φ_i^{-1} has only simple branch points at a_i, that

$$\frac{4r}{3} - 4 \leqq r,$$

or $r \leqq 12$. If $r = 12$, φ_i^{-1} has six simple branch points at a_i. Hence, if φ_i^{-1} has an A-point at a_i, it has six, and if it has one at b_i (or at c_i), it has four. If $r = 9$, there is a simple point and four simple branch points at a_i. The simple point must be the only A-point at a_i, else there would be five. But if there were an A-point at b_i (or at c_i) there would be three, which is impossible because of the A-point at a_i. When $r = 6$, there are two simple points at a_i, and the impossibility follows as in the preceding cases. In the case in which $r \equiv 0$, mod 3, and in which there are simple points at b_i, or c_i, we see that there must be just three simple points, either at b_i or at c_i. This would require that the index of φ_i^{-1} at a_i be $2r/3$, which is impossible, since φ_i^{-1} has only simple branch points at a_i.

Thus, when $\gamma = 3$, we have $r \equiv 1$, mod 3, so that there is a simple point at b_i and at c_i. The sum of the indices at b_i and c_i is $4(r-1)/3$, so that the index at a_i is $2(r-1)/3$. Then

$$\frac{4(r-1)}{3} \leqq r,$$

so that $r = 4$. There are two simple branch points at a_i.

In the case of $r = 4$, we must have $s = 4$, or $s = 3$. The branches of ψ_i^{-1} must undergo at a_i a substitution of order 2, else φ_i^{-1} would have two A-points at a_i, in addition to the two simple points at b_i and c_i. Then we cannot have $s = 4$, for in that case, either none or two of α_{i+1} etc. would equal 2, according as ψ_i^{-1} had none or two simple points at a_i. Thus $s = 3$, and ψ_i^{-1} has a simple point and a simple branch point at a_i. At either b_i or c_i, ψ_i^{-1} must have a branch point of order 2, else φ_i^{-1} would have four A-points. Let it be at b_i. Then ψ_i^{-1} has a simple point and a simple branch point at c_i.

We show now that the values 4 and 5 are impossible for γ_i, so that $\gamma_i = 6$ when $r > 4$.

We may suppose that $\alpha_{i+1} = 2$, $\beta_{i+1} = 3$. We shall show first that if γ_i were 4 or 5, γ_{i+1} would equal γ_i. This will follow as soon as we show that $\gamma_{i+1} \neq 3$, for, from the way in which α_{i+1} etc. depend upon α_i etc., it is clear that γ_{i+1} cannot exceed 4 if $\gamma_i = 4$, and that γ_{i+1} cannot be 4, or more than 5, if $\gamma_i = 5$.

That $\gamma_{i+1} \neq 3$ when $r > 4$ was proved above. Let $r = 4$. We have to consider the possibility of $\gamma_i = 4$. In that case, φ_i^{-1} must have two simple points at a_i and one at b_i. That φ_i^{-1} may have no other A-points, it is necessary that $\beta_i' = 3$. Hence, as $s \leqq 4$, ψ_i^{-1} cannot have more than one A-point at b_i. Thus there cannot be two of the numbers α_{i+1} etc. which equal 3, and as $\beta_{i+1} = 3$, $\gamma_{i+1} \neq 3$. This finishes the proof that $\gamma_{i+1} = \gamma$.

When $\gamma_i = 5$, we see directly that for α_{i+1} etc. to equal 2, 3, 5, respectively, ψ_i^{-1} must have one and only one A-point at each of a_i, etc. Also when $\gamma = 4$, ψ_i^{-1} must have one and only one point at c_i whose order is prime to 4, so that s is odd, and ψ_i^{-1} must also have an A-point at a_i.

Thus, if γ_i is either 4 or 5, ψ_i^{-1} has a single A-point at each of a_i, etc. We shall show that these A-points are simple points, and that α_i' etc. are respectively equal to 2, 3 and γ_i.

First, we cannot have $\alpha_i' = \beta_i' = \gamma_i' = 3$, for the work of the preceding section shows that ψ_i^{-1} would have three simple points at a_i, etc., and ψ_i^{-1} would have other A-points, either at a_i, or at c_i. Hence, by (9), one of α_i', etc., is 2. Now, neither β_i' nor γ_i' can equal 2, else ψ_i^{-1} would have more than one A-point at b_i or c_i respectively. Thus, $\alpha_i' = 2$, so that ψ_i^{-1} has a simple point at a_i. Now ψ_i^{-1} must have more than one point at c_i, else, since it has only simple branch points at a_i, it would have only simple branch points at b_i, and β_i' would equal 2. As ψ_i^{-1} has just one A-point at c_i, the orders of all of its points but one at c_i must be multiples of γ_i. Since $\beta_i' > 2$, we must have $\gamma_i' < 2\gamma_i$, for $\gamma_i > 3$, and the sum of the reciprocals of α_i' etc. is at least unity. Thus $\gamma_i' = \gamma$, and there is a simple point at c_i. By similar reasoning, we find that $\beta_i' = 3$, and that there is a simple point at b_i.

Now, the sum of the indices of ψ_i^{-1} is

$$\frac{s-1}{2} + \frac{2(s-1)}{3} + \frac{(\gamma_i - 1)(s-1)}{\gamma_i},$$

a quantity inferior to $2s-2$, if γ_i is 4 or 5.

We have proved that $\gamma_i = 6$ when it is not 3.

Before discussing the surfaces of φ_i^{-1} and ψ_i^{-1}, for $\gamma_i = 6$, we shall show that $\alpha_i' = 2$ and $\beta_i' = 3$. If α_i' is not two, the only way in which φ_i^{-1} can have fewer than four A-points at a_i is for φ_i^{-1} to have just six branches, which are permuted in three pairs at a_i. This implies, however, since $\gamma_i = 6$, that φ_i^{-1} has simple points at b_i. Hence $\alpha_i' = 2$. Again, if $\beta_i' \neq 3$, every point of φ_i^{-1}

at b_i is an A-point. Hence, $r < 10$, and $r \neq 8$. Also, if r is odd, φ_i^{-1} has a simple point at a_i, so that r cannot be 9 or 7. Finally, if r be 6, φ_i^{-1} must have no simple points at b_i, and this requires that φ_i^{-1} have four simple points at a_i. Hence $\beta_i' = 3$.

In what follows, we shall not use the condition that $r \geqq s$. Thus all results obtained for φ_i^{-1} will hold for ψ_i^{-1}, when $\gamma_i' = 6$.

We shall show first that we cannot have $r \equiv 2$, mod 3. We may assume that α_{i+1}, β_{i+1}, γ_{i+1} are 2, 3 and 6 respectively. We know then that $\alpha_{i+1}' = 2$, $\beta_{i+1}' = 3$. Now, as φ_i^{-1} has two simple points at b_i if $r \equiv 2$, mod 3, we must have $\gamma_{i+1}' = 3$. But if $r \equiv 2$, mod 3, not all of the points of φ_{i+1}^{-1} at c_{i+1} can have orders which are multiples of 3. This fact, together with the fact that φ_{i+1}^{-1} has two simple points at b_{i+1}, shows that $\alpha_{i+2}' = \beta_{i+2}' = \gamma_{i+2}' = 3$, whereas we must have $\alpha_{i+2}' = 2$.

Thus, $r \equiv 0, 1, 3$ or 4, mod 6. We examine the surface of φ_i^{-1} for each of these cases.

First, suppose that $r \equiv 0$, mod 6. Every point of φ_i^{-1} at b_i is of order 3. Hence the index of φ_i^{-1} at b_i is $2r/3$. At a_i, φ_i^{-1} will have either none or two simple points. Hence its index at a_i is either $r/2$ or $r/2 - 1$. Thus the index of φ_i^{-1} at c_i is either $5r/6 - 2$ or $5r/6 - 1$; we shall see that it has the former value, so that there are no simple points at a_i.

Suppose that φ_i^{-1} has, at c_i, w points of order 6, x points of order 3, y points of order 2 and z simple points. Then

$$6w + 3x + 2y + z = r. \tag{27}$$

If the index at c were $5r/6 - 1$, we would have

$$5w + 2x + y = \frac{5r}{6} - 1. \tag{28}$$

Multiplying through by 6/5 in (28), and subtracting the result from (27), we find

$$\frac{3x}{5} + \frac{4y}{5} + z = \frac{6}{5}.$$

The only solution of this equation in positive integers is $x = 2$, $y = 0$, $z = 0$. This means that φ_i^{-1} must have two points of order 3 at c_i, and that all its other points at c_i, if such exist, are of order 6. If γ_i' were not 3. φ_i^{-1} would have at least two A-points at c_i, in addition to the two simple points at a_i. But if $\gamma_i' = 3$, φ_i^{-1} will have no A-points other than the two at a_i. Thus there are no simple points at a_i, and we have

$$5w + 2x + y = \frac{5r}{6} - 2,$$

so that

$$\frac{3x}{5} + \frac{4y}{5} + z = \frac{12}{5}.$$

The solution $x = 4$, $y = z = 0$ leads to an absurdity by the argument just given. The only other solution is $x = y = z = 1$.

Hence, when $r \equiv 0$, mod 6, all branches of φ_i^{-1} must be permuted in pairs at a_i, and in triples at b_i. At c_i, there is a simple point, a point of order 2, and one of order 3. If there are other points at c_i, they are of order 6.

We prove now that $\gamma_i' = 6$. By (9), $2 \leqq \gamma_i' \leqq 6$. If γ_i' were 2, 3 or 4, φ_{i+1}^{-1} would have a critical point with at least three simple points which would arise from the critical point of φ_i^{-1} at c_i. This is impossible, for the critical points of φ_{i+1}^{-1} are evidently similar to those of φ_i^{-1}. If γ_i' were 5, α_{i+1}' etc. would have the impossible values 5, 5, 5. Thus, $\gamma_i' = 6$.

In the case of $r \equiv 1$, mod 6, φ_i^{-1} has one simple point at a_i, and one at b_i. We see as above, only with less trouble, that φ_i^{-1} has one simple point at c_i, and that its other points at c_i are of order 6; also that $\gamma_i' = 2, 3$, or 6, according as $s = 3$, $s = 4$ or $s > 4$.

Similarly, when $r \equiv 3$, mod 6, φ_i^{-1} has a single simple point at a_i, and none at b_i. At c_i, it has a simple point, a simple branch point, and all its other points are of order 6. As before, γ_i' is a divisor of 6. It is impossible for s to be 3. When $s = 4$, $\gamma_i' = 3$, and when $s > 4$, $\gamma_i' = 6$.

We suppose finally that $r \equiv 4$, mod 6. At b_i, φ_i^{-1} must have just one simple point, and its index is $2(r-1)/3$. The index at a_i would be $r/2 - 1$ if there were two simple points at a_i, otherwise $r/2$. Hence the index of φ_i^{-1} at c_i is either

$$\frac{5r}{6} - \frac{1}{3} \quad \text{or} \quad \frac{5r}{6} - \frac{4}{3};$$

we shall show that the index has the latter value, so that φ_i^{-1} has no simple points at a_i.

Suppose that the index had the first value. We would have

$$5w+2x+y=\frac{5r}{6}-\frac{1}{3},$$

and using (27), we find

$$\frac{3x}{5}+\frac{4y}{5}+z=\frac{2}{5},$$

which has no solutions. Hence the second value is the true one, and

$$\frac{3x}{5}+\frac{4y}{5}+z=\frac{8}{5}.$$

This equation has the solutions $x=0$, $y=2$, $z=0$, and $x=1$, $y=0$, $z=1$. We show that the solution $y=2$ is impossible. First, γ_i' is a divisor of 6, else φ_i^{-1} would have at least three A-points at c_i in addition to the simple point at b_i. But as all the points of φ_i^{-1} at c_i would be of even order if $y=2$, the only way for α_{i+1}' to equal 2 would be for γ_i' to be divisible by 4. Hence $y \neq 2$. Thus, φ_i^{-1} has one simple point and one point of order 3 at c_i, and its other points at c are of order 6.

If $s=3$, $\gamma_i'=2$. Also, $s=4$ is impossible. Finally, $\gamma_i'=6$ when $s>3$.

We have already called attention to the fact that when $\gamma_i'=6$ the above discussion of the surface of φ_i^{-1} applies also to the surface of ψ_i^{-1}.

The surfaces of φ_i^{-1} and ψ_i^{-1} being recognized, we can use the methods of the foregoing sections to work back to the surfaces of Φ^{-1} and Ψ^{-1}. There are, indeed, several cases to be examined, and diophantine equations like those used above must be employed in discussing the surface of σ_i^{-1}, but no new ideas have to be introduced.

We find that the degrees of Φ^{-1} and Ψ^{-1} are congruent to $0, 1, 3$ or 4, mod 6, and that the inverse of each has three critical points, called below a_0, b_0, c_0. We shall describe Ψ; similar remarks will apply to Φ.

If $n \equiv 0$, mod 6, the branches of Ψ^{-1} are all permuted in pairs at a_0, and all in triples at b_0. At c_0, Ψ^{-1} has one uniform branch with value c_0, one simple branch point at which $\Psi^{-1}=b_0$, and one branch point of order 2 at which $\Psi^{-1}=a_0$. If $n>6$, the remaining branches are permuted in sixes at c_0.

If $n \equiv 1$, mod 6, Ψ^{-1} has one uniform branch at a_0 whose value is a_0, and the other branches of Ψ^{-1} are permuted in pairs at a_0; at b_0, there is a uniform branch whose value is b_0, and the other branches are permuted in triples; at c_0, there is one uniform branch whose value is c_0, while the other branches are permuted in sixes.

If $n \equiv 3$, mod 6, there is one uniform branch at a_0 whose value is a_0; at c_0 there is one uniform branch whose value is c_0, and one simple branch point for which $\Psi^{-1} = b_0$. If $n > 3$, the other branches are permuted in sixes at c_0.

If $n \equiv 4$, mod 6, Ψ^{-1} has one uniform branch at b_0 whose value is b_0. At c_0, Ψ^{-1} has one uniform branch whose value is c_0, and one branch point of order 2 at which $\Psi^{-1} = a_0$. Also if $n > 4$, the other branches are permuted in sixes at c_0.

To identify Φ and Ψ, we use the function $\wp(z \mid \omega, e^{\pi i/3}\omega)$ of the preceding section. Putting $\wp\omega_1 = e_1$, we have, from the homogeneity formulas,

$$e_2 = \wp\omega_2 = e^{\frac{2\pi i}{3}} e_1, \qquad e_3 = \wp\omega_3 = e^{\frac{4\pi i}{3}} e_1.$$

The values e_i cannot be zero, else $\wp z$ would have six zeros in a parallelogram. Hence, as $\wp' z$ vanishes only when z is a half-period, $\wp z$ vanishes only once wherever it vanishes.

We consider now the function $\wp^3 z$. Wherever it assumes the values e_1^3, 0, and ∞, it assumes them 2, 3 and 6 times respectively. There are no other multiple values.

We take a linear $\lambda(z)$ such that

$$\lambda(a_0) = e_1^3, \qquad \lambda(b_0) = 0, \qquad \lambda(c_0) = \infty,$$

and introduce Φ_1 and Ψ_1. We find

$$\wp^3(az+b) = \Phi_1(\wp^3 z), \qquad \wp^3(cz+d) = \Psi_1(\wp^3 z),$$

where

$$2a\omega \equiv 0, \qquad 2c\omega \equiv 0 \qquad (\text{modd}\, 2\omega,\, 2e^{\frac{\pi i}{3}}\omega);$$

$$b\left(1-e^{\frac{\pi i}{3}}\right) \equiv 0, \qquad d\left(1-e^{\frac{\pi i}{3}}\right) \equiv 0 \qquad (\text{modd}\, 2\omega,\, 2e^{\frac{\pi i}{3}}\omega);$$

$$(a-1)d \equiv (c-1)b \qquad (\text{modd}\, 2\omega,\, 2e^{\frac{\pi i}{3}}\omega).$$

The second congruence is found from the equation $\wp^3 e^{\pi i/3} z = \wp^3 z$.

X. Permutable functions with a common iterate

The results of the preceding section are based upon the assumption that every function of the sequence (C) is of degree less than m and less than n; that is, that for no i is one of the functions $\sigma_i \varphi_i$, $\sigma_i \psi_i$ a rational function of the other. The removal of this assumption will lead to a new class of permutable pairs of functions.

It will be convenient, in what follows, to represent $\varphi_i \sigma_i$ by Φ_i and $\psi_i \sigma_i$ by Ψ_i. The preceding sections deal with the sequences

$$(29) \qquad \begin{array}{l} \Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_i, \ldots, \\ \Psi_0, \Psi_1, \Psi_2, \ldots, \Psi_i, \ldots. \end{array}$$

Let us suppose now that for $i \geqq i_0$ (this i_0 is not to be confused with the i_0 of § III), one of the functions Φ_i, Ψ_i is a rational function of the other. It fixes the ideas to suppose that i_0 is the smallest number of this type. If we assume that $m \geqq n$, we will be sure that it is Φ_{i_0} which is a rational function of Ψ_{i_0}. Let $\Phi_{i_0} = \beta_0 \Psi_{i_0}$. The permutability of Φ_{i_0} and Ψ_{i_0} gives

$$\beta_0 \Psi_{i_0} \Psi_{i_0} = \Psi_{i_0} \beta_0 \Psi_{i_0},$$

or

$$\beta_0 \Psi_{i_0} = \Psi_{i_0} \beta_0,$$

so that Ψ_{i_0} is permutable with β_0.

We distinguish the following two cases:

(*a*) β_0 is of degree greater than unity;

(*b*) β_0 is linear.

Suppose that we have met Case (*a*). It will be convenient to replace the pair of symbols Ψ_{i_0} and β_0 by the pair Φ_{10} and Ψ_{10}, and we suppose the new symbols to replace the old in such a way that the degree of Φ_{10} is not less than that of Ψ_{10}. We have thus

$$\Phi_{i_0} = \Phi_{10} \Psi_{10} = \Psi_{10} \Phi_{10},$$

and Ψ_{i_0} is either Φ_{10} or Ψ_{10}.

We deal now with Φ_{10} and Ψ_{10} exactly as we dealt originally with Φ_0 and Ψ_0, and obtain the sequences

$$\Phi_{10},\ \Phi_{11},\ \Phi_{12},\ \ldots\ \Phi_{1i},\ \ldots$$

$$\Psi_{10},\ \Psi_{11},\ \Psi_{12},\ \ldots\ \Psi_{1i},\ \ldots$$

The following two cases are here possible:

(a) There is no i such that Φ_{1i} is a rational function of Ψ_{1i};

(b) For $i \geqq i_1$, Φ_{1i} is a rational function of Ψ_{1i}.

If Case (b) is at hand, we deal with Φ_{1i_1} and Ψ_{1i_1} exactly as we treated Φ_{i_0} and Ψ_{i_0} above.

The process we are employing leads finally to two sequences of permutable functions, rational and non-linear,

$$(30)\qquad \begin{array}{l} \Phi_{p0},\ \Phi_{p1},\ \Phi_{p2},\ \ldots,\ \Phi_{pi},\ \ldots, \\ \Psi_{p0},\ \Psi_{p1},\ \Psi_{p2},\ \ldots,\ \Psi_{pi},\ \ldots, \end{array}$$

the common degree of the functions in the first sequence being at least equal to that of the functions in the second, and the two sequences having one of the two following properties:

(I) There is no i such that Φ_{pi} is a rational function of Ψ_{pi};

(II) For $i \geqq i_p$, Φ_{pi} is a *linear* function of Ψ_{pi}.

Case (I), which does not yield any new types of functions, is quickly disposed of.

We know that Φ_{p0} and Ψ_{p0} come from one of the several types of multiplication theorems discussed in the preceding sections. To take an example which is typical, suppose that the periodic function involved is $\cos z$. Then the inverses of the three functions

$$\Phi_{p0}, \qquad \Psi_{p0}, \qquad \Phi_{p0}\,\Psi_{p0}$$

have no critical points other than certain three points a_0, b_0, c_0, at the first two of which their branches are permuted in pairs, except that there are two places on the surface of each inverse at a_0 and b_0 where each inverse is uniform and assumes the values a_0 and b_0; at c_0, the branches of each inverse are permuted in a single cycle, and the single value of each inverse is c_0. Now, since

$$\Phi_{p-1,\,i_{p-1}} = \Phi_{p0}\,\Psi_{p0},$$

and since $\Psi_{p-1,i_{p-1}}$ is either Φ_{p0} or Ψ_{p0}, we can work back to $\Phi_{p-1,0}$, $\Psi_{p-1,0}$, and show by the familiar process that these last two functions come from the multiplication formulas for $\cos z$. Continuing in this fashion, we find that Φ_0 and Ψ_0 are also given by the multiplication formulas for $\cos z$.

An examination of all other possibilities shows similarly that every pair of permutable functions which comes under Case (I) is given by one of the multiplication formulas of the preceding sections.

We take now Case (II), in which Φ_{pi_p} is a linear function of Ψ_{pi_p}. For brevity, we represent these two functions by Φ_∞ and Ψ_∞, respectively. We have

$$\Phi_\infty = \beta\Psi_\infty,$$

where $\beta(z)$ is linear.

As above, Ψ_∞ and β are permutable. We have thus to determine the circumstances under which a rational function of degree greater than unity is permutable with a linear function. This question has been solved by Julia,* but it will not hurt to give a brief treatment of it here.

The linear function $\beta(z)$ has either two fixed points or one. In the first case, if $\lambda(z)$ is a linear function which carries the fixed points to 0 and ∞ respectively, $\lambda\beta\lambda^{-1}$ will be of the form ϵz. In the second case, if $\lambda(z)$ carries the single fixed point to ∞, $\lambda\beta\lambda^{-1}$ will be of the form $z+h$. Thus, if Φ_0 and Ψ_0 and all of the functions of the several sequences are transformed with λ^{-1}, we may suppose that $\beta(z)$ is either ϵz or $z+h$.

If $\beta(z)$ were of the form $z+h$, with $h \neq 0$, we would have

$$\Psi_\infty(z+h) = \Psi_\infty(z)+h,$$

so that, indicating differentiation with an accent, we find

$$\Psi'_\infty(z+h) = \Psi'_\infty(z).$$

Hence $\Psi'_\infty(z)$ would be periodic, and, being rational, would be a constant. This would require that $\Psi_\infty(z)$ be linear.

Thus $\beta(z)$ must have two fixed points, and

$$\Psi_\infty(\epsilon z) = \epsilon\Psi_\infty(z). \tag{31}$$

* Loc. cit., p. 177.

Differentiating, we find

$$\Psi'_\infty(\varepsilon z) = \Psi'_\infty(z).$$

If ε were not a root of unity, $\Psi'_\infty(z)$ would assume, for each of the distinct points $\varepsilon^r z_1 (z_1 \neq 0, \infty; r = 1, 2, \ldots)$, the same value as at z_1. Hence, $\Psi'_\infty(z)$, being rational, would be a constant, and $\Psi_\infty(z)$ would be linear.

Thus, ε is a root of unity, let us say, a primitive rth root of unity. We read directly from (31) that $\Psi_\infty(z)/z$ is a rational function of z^r, that is,

$$\Psi_\infty(z) = zR(z^r),$$

where $R(z)$ is a rational function.

Referring now to the definitions, given in the introduction, of the operations of the first and second types, we see immediately that *if the pair of non-linear permutable functions* $\Phi(z)$ *and* $\Psi(z)$ *do not come from the multiplication theorems of the periodic functions, there exists a linear* $\lambda(z)$ *such that* $\lambda\Phi\lambda^{-1}$ *and* $\lambda\Psi\lambda^{-1}$ *can be obtained by repeated operations of the first and second types, starting from a pair of functions*

$$zR(z^r), \qquad \varepsilon z R(z^r),$$

where $R(z)$ *is rational, and where* ε *is a primitive rth root of unity.*

As we do not determine all cases in which operations of the first type are possible, the process just described can certainly not be accepted as furnishing a neat characterization of the pairs of functions which come under Case (II). Nevertheless, we shall progress much further in the study of this case. In particular, we shall settle completely the case in which $\Phi(z)$ and $\Psi(z)$ are polynomials.

We begin by proving that *if the pair of permutable functions* $\Phi(z)$ *and* $\Psi(z)$ *come under Case* (II), *in particular, if* $\Phi(z)$ *and* $\Psi(z)$ *do not come from the multiplication theorems of the periodic functions, some iterate of* $\Phi(z)$ *is identical with some iterate of* $\Psi(z)$.

We have shown that

$$\Phi_{pi_p} = \varepsilon\Psi_{pi_p}.$$

We denote the nth iterate of any function $F(z)$ by $F^{(n)}(z)$. As Ψ_{pi_p} is permutable with εz, we have

$$\Phi^{(n)}_{pi_p} = \varepsilon^n \Psi^{(n)}_{pi_p},$$

and, in particular, since $\epsilon^r = 1$,

$$\Phi_{pi_p}^{(r)} = \Psi_{pi_p}^{(r)}. \tag{32}$$

We have regard now to the manner in which the two functions of (32) are obtained from those which precede them in (30). We write

$$\begin{aligned} \Phi_{p,i_p-1} &= \varphi\sigma, \qquad \Psi_{p,i_p-1} = \psi\sigma, \\ \Phi_{pi_p} &= \sigma\varphi, \qquad \Psi_{pi_p} = \sigma\psi, \end{aligned} \tag{33}$$

our failure to attach subscripts to σ, φ and ψ causes no confusion. Thus (32) may be written

$$\sigma\varphi \cdots \sigma\varphi = \sigma\psi \cdots \sigma\psi$$

so that, operating on both sides of the last equation with φ, and replacing z by $\sigma(z)$, we have

$$\varphi\sigma\varphi \cdots \sigma\varphi\sigma = \varphi\sigma\psi \cdots \sigma\psi\sigma.$$

As $\varphi\sigma$ and $\psi\sigma$ are permutable, we find

$$\varphi\sigma\varphi \cdots \sigma\varphi\sigma = \psi \cdots \sigma\psi\sigma\varphi\sigma.$$

Removing $\varphi\sigma$ from the beginning of each member of this equation, we have

$$\varphi\sigma \cdots \varphi\sigma = \psi\sigma \cdots \psi\sigma,$$

that is,

$$\Phi_{p,i_p-1}^{(r)} = \Psi_{p,i_p-1}^{(r)}.$$

Continuing thus, we see that the rth iterates of Φ_{p0} and Ψ_{p0} are equal. Now, since

$$\Phi_{p-1,i_{p-1}} = \Phi_{p0}\Psi_{p0}$$

we see directly that

$$\Phi^{(r)}_{p-1,i_{p-1}} = \Psi^{(2r)}_{p-1,i_{p-1}}.$$

There is nothing to prevent us from working back through the sequence which precedes (30) as we did through (30), and thence onward through the earlier sequences. We find thus that some iterate of $\Phi(z)$ is identical with some iterate of $\Psi(z)$.

We turn for a while to the case in which $\Phi(z)$ and $\Psi(z)$ are polynomials. In a paper published a few years ago,* we determined all pairs of polynomials which have an iterate in common, and the results there produced could be used to settle quickly the problem now before us. However, we shall gain a better insight into the nature of the fractional permutable functions which have an iterate in common by studying the polynomial case from the group-theoretic point of view of the present paper, and of our paper *Prime and composite polynomials.*

From a result stated at the bottom of page 54 of the paper just mentioned, it follows that, if $\Phi(z)$ and $\Psi(z)$ are polynomials, we may suppose that the functions in the sequences (29) to (30) inclusive are all polynomials.

We prove now that when $\Phi(z)$ and $\Psi(z)$ are polynomials, then, in (30), Φ_{p0} is a linear function of Ψ_{p0}. Suppose that this is not so, and that the first pair of functions in (30) which are linear functions of each other are Φ_{pi_p} and Ψ_{pi_p}, where $i_p > 0$. We have already seen that if $\Phi(z)$ and $\Psi(z)$ are subjected to a suitable linear transformation (in this case integral), we will have

$$\Phi_{pi_p} = \varepsilon\Psi_{pi_p},$$

or, by (33),

$$\Phi_{pi_p} = \sigma\varphi = \varepsilon\sigma\psi. \tag{34}$$

Now the algorithm which produces the sequence (30) supposes that no rational $\beta(z)$ of degree greater than 1 exists, such that $\varphi = \zeta\beta$, $\psi = \xi\beta$. We have shown, however, (loc. cit., p. 56) that if Φ_{pi_p} has two decompositions of the types shown in (34), in which φ and ψ are of the same degree, φ must be a linear function of ψ. This completes the proof that Φ_{p0} is a linear function of Ψ_{p0}.

* These Transactions, vol. 21 (1920), p. 313.

We may suppose, thus, that

$$\Phi_{p0} = \varepsilon z R(z^r), \qquad \Psi_{p0} = z R(z^r),$$

where ε is an rth root of unity, and where $R(z)$ is a polynomial.*

Hence

$$\Phi_{p-1,i_{p-1}} = \varepsilon \Psi_{p0}^{(2)}, \qquad \Psi_{p-1,i_{p-1}} = \varepsilon_1 \Psi_{p0}, \tag{35}$$

where ε_1, if not equal to ε, is unity.

We shall show now that $\Phi_{p-1,0}$ is a rational function of $\Psi_{p-1,0}$, so that these two functions may be considered to be given by (35). If this were not the case, we would have

$$\Phi_{p-1,i_{p-1}} = \sigma \varphi, \qquad \Psi_{p-1,i_{p-1}} = \sigma \psi, \tag{36}$$

where no non-linear $\beta(z)$ exists such that $\varphi = \sigma\beta$, $\psi = \xi\beta$. From (35), and the second equation of (36), we find

$$\Phi_{p-1,i_{p-1}} = \varepsilon \varepsilon_1^{-1} \Psi_{p0} \Psi_{p-1,i_{p-1}} = \varepsilon \varepsilon_1^{-1} \Psi_{p0} \sigma \psi. \tag{37}$$

Thus, by (37) and the first equation of (36), φ and ψ would determine systems of imprimitivity of the group of the inverse of $\Phi_{p-1,i_{p-1}}$ with not more than one letter in common. For this it would be necessary (loc. cit., p. 57), that the degrees of φ and ψ be prime to each other. This produces the contradiction that the degree of $\Phi_{p-1,i_{p-1}}$ is not divisible by that of $\Psi_{p-1,i_{p-1}}$. Hence, we have

$$\Phi_{p-2,i_{p-2}} = \varepsilon_2 \Psi_{p0}^{(3)}, \qquad \Psi_{p-2,i_{p-2}} = \varepsilon_3 \Psi_{p0}^{(j)},$$

where ε_2 and ε_3 are rth roots of unity, and where j is 1 or 2.

Continuing thus, we find that *if $\Phi(z)$ and $\Psi(z)$ are permutable polynomials (non-linear), which do not come from the multiplication formulas of e^z or* cos z, *there exist a linear $\lambda(z)$, and a polynomial*

$$G(z) = z R(z^r),$$

* The case of $r = 1$ does not require a separate statement, for we may suppose, using a suitable linear transformation, that all polynomials met are divisible by z.

such that

$$\Phi = \lambda^{-1}(\varepsilon_1 G^{(\mu)} \lambda), \qquad \Psi = \lambda^{-1}(\varepsilon_2 G^{(\nu)} \lambda)$$

where ε_1 *and* ε_2 *are rth roots of unity.*

Furthermore, this necessary condition for permutability is immediately seen to be sufficient. In fact, if $R(z)$ is any rational function, integral or fractional, the above formulas will give a pair of permutable rational functions.

When we seek explicit formulas for the permutable pairs of fractional functions which do not come from the multiplication theorems of the periodic functions, things do not go through smoothly. For instance, it is not necessary that one of the functions Φ and Ψ should be a rational function of the other, as is always the case for polynomials.

We shall give an example of such a case. Let

$$\varphi(z) = \frac{\varepsilon^2 z^2 + 2}{\varepsilon z + 1}, \qquad \psi(z) = \frac{z^2 + 2}{z + 1}, \qquad \sigma(z) = \frac{z^2 - 4}{z - 1},$$

where ε is a primitive third root of unity. We shall see below that $\Phi = \varphi\sigma$ and $\Psi = \psi\sigma$ are permutable. We observe at present that Φ is not a linear function of Ψ; if it were, φ would be a linear function of ψ, which is not so, because ψ is infinite for $z = -1$ and $z = \infty$, whereas φ is infinite for $z = \infty$, but not for $z = -1$. We have

$$\sigma\varphi = \varepsilon z \frac{z^3 - 8}{z^3 - 1}, \qquad \sigma\psi = z \frac{z^3 - 8}{z^3 - 1}.$$

As $\varphi(z) = \psi(\varepsilon z)$, and as $\sigma\psi(\varepsilon z) = \varepsilon\sigma\psi(z)$, we have

$$\varphi\sigma\psi\sigma = \psi(\varepsilon\sigma\psi\sigma) = \psi\sigma\psi(\varepsilon\sigma) = \psi\sigma\varphi\sigma.$$

This means that Φ and Ψ, which, we repeat, are not linear functions of each other, are permutable.

It can be shown that Φ and Ψ do not come from the multiplication theorems of the periodic functions. We shall escape the calculations connected with this question by modifying the example. Let $\beta(z)$ be any rational function such that $\beta(\varepsilon z) = \varepsilon\beta(z)$, where ε is a primitive third root of unity. We consider the functions

$$\Phi = \varphi\beta\sigma, \qquad \Psi = \psi\beta\sigma,$$

where σ, φ and ψ are the three functions of the second degree used above. As above, Φ is not a linear function of Ψ. Also,

$$\varphi\beta\sigma\psi\beta\sigma = \psi(\varepsilon\beta\sigma\psi\beta\sigma) = \psi\beta\sigma\psi(\varepsilon\beta\sigma) = \psi\beta\sigma\varphi\beta\sigma,$$

so that Φ and Ψ are permutable.

It is easily seen, since $\beta(z)$ is of a very general type, that, by suitably choosing $\beta(z)$, we can make the critical points of Φ^{-1} and of Ψ^{-1} numerous and complicated at pleasure. This means that Φ and Ψ cannot come from the multiplication theorems of the periodic functions, for, if they did, their inverses would have at most four critical points.

In the above example, Φ and Ψ have the same third iterate.

Concerning the pairs of permutable fractional functions which come neither from the multiplication theorems of the periodic functions, nor from the iteration of a function, the only information we have, in addition to the fact that the functions of the pair have an iterate in common, is that contained in the statement on page 443. We think that the example given above makes it conceivable that no great order may reign in this class.

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